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SOME REMARKS ON THE CLASSIFICATION OF THE WEYL CONFORMAL TENSOR IN 4-DIMENSIONAL MANIFOLDS OF NEUTRAL SIGNATURE

BY

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Abstract. This paper presents a brief discussion of the algebraic classification of the Weyl conformal tensor on a $4-$dimensional manifold with metric g of neutral signature $(+,+,-,-)$. The classification is algebraically similar to the well-known Petrov classification in the Lorentz case and the various algebraic types and corresponding canonical forms are obtained. Further details on principal, totally null $2-$spaces and null directions similar to those of L. Bel in the Lorentz case are described.

Keywords: Weyl tensor classification; neutral signature; algebraic structures.

1. Introduction

Let M be a $4-$dimensional manifold with smooth metric of neutral signature $(+,+,-,-)$ and let C be the Weyl conformal tensor for (M,g). The idea is to provide an algebraic classification of C similar to that given by Petrov in the Lorentz case. The discussion here is brief and more details will be given elsewhere (Hall, 2017). After this work was completed the author was

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informed that ideas similar to some of those reported here have been given in (Law, 1991; Law, 2006; Batista, 2013; Ortaggio, 2009) and another approach was also presented in (Coley and Hervik, 2010). However, the work here is claimed to be simpler, more structured and to go much further and is more amenable for purposes of calculation.

2. Algebraic and Geometric Preliminaries

At $m \in M$ the tangent space to M, $T_m M$, has a basis x, y, s, t satisfying $x \cdot x = y \cdot y = -s \cdot s = -t \cdot t = 1$ (where $\cdot$ denotes an inner product with respect to $g(m)$) and an associated null basis of (null) vectors l, n, L, N at m given by $\sqrt{2}l = x + t$, $\sqrt{2}n = x - t$, $\sqrt{2}L = y + s$ and $\sqrt{2}N = y - s$ so that $l \cdot n = L \cdot N = 1$ with all other such inner products zero. The space of all $2-$forms (*bivectors*) at m is denoted by $\Lambda_m M$ and is a Lie algebra under matrix commutation. A bivector F has matrix rank either 2 or 4 and, if it is 2, F is called *simple*. A simple bivector may be written in components as $F^{ab} = u^a v^b - v^a u^b$ for $u, v \in T_m M$ and the $2-$dimensional subspace of $T_m M$ spanned by u, v is uniquely determined by F and called the *blade* of F. Now, with $*$ denoting the usual duality operator and for $E \in \Lambda_m M$ one has ${}^*E^* = E$ and one may define the subalgebras $\overset{+}{S}_m \equiv \{E \in \Lambda_m M : \overset{*}{E} = E\}$ and $\overset{-}{S}_m \equiv \{E \in \Lambda_m M : \overset{*}{E} = -E\}$ of $\Lambda_m M$. Each member of $\Lambda_m M$ may be uniquely decomposed into the sum of members of $\overset{+}{S}_m$ and $\overset{-}{S}_m$. One also has a metric P on $\Lambda_m M$ given for $E, E' \in \Lambda_m M$ by $P(E, E') = E^{ab} E'_{ab}$ and this metric has signature $(+,+,-,-,-,-)$. It then follows that if $\overset{+}{E} \in \overset{+}{S}_m$ and $\overset{-}{E} \in \overset{-}{S}_m$, $P(\overset{+}{E}, \overset{-}{E}) = 0$ and P restricts to a metric of Lorentz signature $(+,-,-)$ on each of $\overset{+}{S}_m$ and $\overset{-}{S}_m$. This leads to the Lie algebra product $\Lambda_m M = \overset{+}{S}_m \oplus \overset{-}{S}_m$. Each of $\overset{+}{S}_m$ and $\overset{-}{S}_m$ is Lie-isomorphic to $o(1,2)$ and, of course, $\Lambda_m M$ is Lie-isomorphic to $o(2,2)$. Particularly important simple members of $\overset{+}{S}_m$ and $\overset{-}{S}_m$ are the *totally null bivectors* (and they are the only simple members of $\overset{+}{S}_m$ and $\overset{-}{S}_m$) whose blades are spanned by an orthogonal pair of null members of $T_m M$. Choosing an orientation for $T_m M$ one may then choose a null basis for $T_m M$, as above, and then a basis F, G, H for $\overset{+}{S}_m$ where $F = l \wedge n - L \wedge N$, $G = l \wedge N$

and $H = n \wedge L$ (and similarly $\bar{F} = l \wedge n + L \wedge N$, $\bar{G} = l \wedge L$ and $\bar{H} = n \wedge N$ is a basis for $\bar{S}_m$). In these bases G and H are totally null members of $\overset{+}{S}_m$ and $\bar{G}$ and $\bar{H}$ are totally null members of $\bar{S}_m$.

3. The Weyl Tensor Classification

The Weyl conformal tensor C for (M, g) satisfies ${}^*C = C^*$ and may be decomposed at any $m \in M$ into tensors $\overset{+}{W}$ and $\bar{W}$ as

$$C = \overset{+}{W} + \bar{W} \qquad \overset{+}{W} \equiv \frac{1}{2}(C + {}^*C), \qquad \bar{W} \equiv \frac{1}{2}(C - {}^*C) \tag{1}$$

Thus $\overset{+}{W}{}^* = \overset{+}{W}$ and $\bar{W}^* = -\bar{W}$. Next consider the linear map f on bivectors at m given by $f : E^{ab} \to C^{ab}{}_{cd} E^{cd}$ together with maps $\overset{+}{f}$ and $\bar{f}$ obtained in a similar way from $\overset{+}{W}$ and $\bar{W}$. The subspaces $\overset{+}{S}_m$ and $\bar{S}_m$ are invariant subspaces of f. Now the map $\overset{+}{f} : \overset{+}{S}_m \to \overset{+}{S}_m$ is a linear map on a $3-$dimensional space of Lorentz signature and may be algebraically classified into its Jordan forms (Segre types) and the only types which arise are $\{111\}$ (diagonable over $\mathbb{R}$), $\{1z\bar{z}\}$ (diagonable over $\mathbb{C}$), $\{21\}$ (eigenvalues real) and $\{3\}$ (with eigenvalue zero from the tracefree condition on $\overset{+}{W}$ which follows from that on C). Using the basis for $\overset{+}{S}_m$ given above it can be shown that the above four Jordan types for $\overset{+}{f}$ (that is, for $\overset{+}{W}$) give the following "canonical" forms for $\overset{+}{W}(m)$)

$$\overset{+}{W}_{abcd}(m) = \frac{\rho_1}{2}(G_{ab}H_{cd} + H_{ab}G_{cd} + F_{ab}F_{cd}) + \frac{\rho_2}{2}(G_{ab}G_{cd} \pm H_{ab}H_{cd}) \tag{2}$$

$$\overset{+}{W}_{abcd}(m) = \frac{\rho_1}{2}(G_{ab}H_{cd} + H_{ab}G_{cd} + F_{ab}F_{cd}) \pm G_{ab}G_{cd} \tag{3}$$

$$\overset{+}{W}_{abcd}(m) = (G_{ab}F_{cd} + F_{ab}G_{cd}) \tag{4}$$

for $\rho_1, \rho_2 \in \mathbb{R}$. By analogy with the Petrov classification of $C(m)$ in the Lorentz case (Petrov, 1969) (and cf (Hall, 2004)), call $\overset{+}{W}(m)$ in Eq. (2) type **I** if the eigenvalues are distinct. If two eigenvalues are equal in Eq. (2) (Segre type $\{1(11)\}$) there are two possibilities; first when the resulting eigen-2-space of bivectors has Lorentz signature in $\overset{+}{S}_m$ ($\rho_2 = 0$ in Eq. (2)) and this type is called $\mathbf{D_1}$ and second when this eigen-2-space is Euclidean ($3\rho_1 = \rho_2 \neq 0$ in Eq. (2)) and this type will be labelled $\mathbf{D_2}$. These are the "degenerate" possibilities for type **I**. Similarly call $\overset{+}{W}(m)$ in Eq. (3) type **II** (and call the degenerate case when the eigenvalue $\rho_1 = 0$ type **N**). For Eq. (4) the type is labelled **III**. The degenerate types are thus

$$\begin{aligned} \overset{+}{W}_{abcd}(m) &= \frac{\rho_1}{2}(G_{ab}H_{cd} + H_{ab}G_{cd} + F_{ab}F_{cd}) \\ &= \frac{\rho_1}{2}(2\overset{+}{P}_{abcd} + \frac{3}{2}F_{ab}F_{cd}) \qquad (type\ \mathbf{D_1}; \rho_1 \neq 0) \end{aligned} \tag{5}$$

$$\begin{aligned} \overset{+}{W}_{abcd}(m) &= \frac{\rho_1}{2}(G_{ab}H_{cd} + H_{ab}G_{cd} + F_{ab}F_{cd}) + \frac{3\rho_1}{2}(G_{ab}G_{cd} + H_{ab}H_{cd}) \\ &= -\rho_1(2\overset{+}{P}_{abcd} - \frac{3}{2}K_{ab}K_{cd}) \qquad (type\ \mathbf{D_2}; \rho_1 \neq 0) \end{aligned} \tag{6}$$

$$\overset{+}{W}_{abcd}(m) = \pm G_{ab}G_{cd} \qquad (type\ \mathbf{N}) \tag{7}$$

where $K \equiv G + H$ and $\overset{+}{P}_{abcd} \equiv \frac{1}{2}(G_{ab}H_{cd} + H_{ab}G_{cd} - \frac{1}{2}F_{ab}F_{cd})$. Finally one adds the type **O** at m when $\overset{+}{W}(m) = 0$.

4. Principal Null Directions and Totally Null 2-Spaces

For $\overset{+}{W}(m) \neq 0$ consider the following relationships for a non-zero $k \in T_m M$, a *totally null* bivector $E \in \overset{+}{S}_m$, a non-zero bivector $P \in \overset{+}{S}_m$ *not proportional* to E and satisfying $E_{ab}P^{ab} = 0$, a $1-$form p which is *neither zero nor parallel to* k and real numbers $\alpha, \beta, \gamma, \delta$ with $\delta \neq 0$.

$$(i)\ \overset{+}{W}_{abcd}\,k^b k^d = \alpha k_a k_c, \qquad (ii)\ \overset{+}{W}_{abcd}\,E^{cd} = \beta E_{ab} \tag{8}$$

$$(i)\ \overset{+}{W}_{abcd}\,k^b k^d = k_a p_c + p_a k_c \qquad (ii)\ \overset{+}{W}_{abcd}\,E^{cd} = \gamma E_{ab} + \delta P_{ab} \tag{9}$$

The vector k in Eq. (8(i)) is necessarily null and will be said to span a *repeated principal null direction of* $\overset{+}{W}(m)$ (a *repeated pnd*) (cf (Bel, 2000; Sachs, 1961; Hall, 2004)). The blade of the totally null bivector E in Eq. (8(ii)) will be called a *repeated principal totally null* $2-$*space (a repeated* $2-$*space) of* $\overset{+}{W}(m)$ (and E is an eigenbivector of $\overset{+}{W}(m)$). The vector k in Eq. (9(i)) can be shown to be necessarily null and will be said to span a *general principal null direction of* $\overset{+}{W}(m)$ (a *general pnd*) [and a set of equivalent conditions on k are (i) that $k_{[e}\overset{+}{W}_{a]bc[d}\,k_{f]}k^b k^c = 0$ where square brackets denote the usual skew-symmetrisation of indices, and (ii) that Eq. (8(i)) is false]. Collectively, repeated and general pnds will be referred to simply as *pnds*. The blade of the bivector E in Eq. (9(ii)) will be called a *general principal totally null* $2-$*space* (*a general* $2-$*space*) *of* $\overset{+}{W}(m)$. Collectively, repeated and general such $2-$spaces are called *principal* $2-$ *spaces* of $\overset{+}{W}(m)$. Assuming that $\overset{+}{W}(m) \neq 0$ the following hold;

Lemma 1

(i) There exists $0 \neq k \in T_m M$ such that $\overset{+}{W}_{abcd}\,k^d = 0$ if and only if $\overset{+}{W}(m)$ is type **N**. The vector k spans a repeated pnd and may be *any* non-zero member of the totally null blade of the bivector G in Eq. (7) (and only these). The bivector G is the unique totally null member of $\overset{+}{S}_m$ (up to a scaling) satisfying Eq. (8(ii)) and, in fact, $\beta = 0$.

(ii) There exists $0 \neq k \in T_m M$ such that $\overset{+}{W}_{abcd}\,k^b k^d = 0$ if and only if $\overset{+}{W}(m)$ is type **N** or **III**. Again k spans a repeated pnd and may be *any* non-zero member of the totally null blade of the bivector G in Eq. (7) or Eq. (4) (and only these). The bivector G is the unique totally null member of $\overset{+}{S}_m$ (up to a scaling) satisfying Eq. (8(ii)) and, in fact, $\beta = 0$.

(*iii*) There exists $0 \neq k \in T_m M$ such that $\overset{+}{W}_{abcd} k^b k^d = \alpha k_a k_c$ with $0 \neq \alpha \in \mathbb{R}$ if and only if $\overset{+}{W}(m)$ is type **II** or $\mathbf{D_1}$. Again k spans a repeated pnd and may be *any* non-zero member of the totally null blade of the bivector G in Eq. (3) for type **II** (and only these), or any member of the totally null blades of G and H in Eq. (5) for $\mathbf{D_1}$ (and only these). The bivectors G (for type **II**) and G and H (for type $\mathbf{D_1}$) are the unique totally null member(s) of $\overset{+}{S}_m$ (up to a scaling) satisfying Eq. (8(*ii*)) and in all cases $\beta \neq 0 \neq \alpha$ with the same β arising for both G and H and the same α for the associated pnds in type $\mathbf{D_1}$.

(*iv*) If there exists $0 \neq k \in T_m M$ such that Eq. (9(*i*)) holds then k spans a general pnd and may be *any* member of the totally null blade of a bivector $E \in \overset{+}{S}_m$ satisfying Eq. (9(*ii*)). The non-zero members of the blade of any totally null $E \in \overset{+}{S}_m$ satisfying Eq. (9(*ii*)) span general pnds.

Thus finding repeated pnds for $\overset{+}{W}(m)$ amounts to finding its totally null eigenbivectors E as in Eq. (8(*ii*)). If such an eigenbivector exists either it is unique (up to a scaling) and then the type of $\overset{+}{W}(m)$ is **N**, **III** ($\beta = 0 = \alpha$ in Eq. (8)) or **II** ($\beta \neq 0 \neq \alpha$ in Eq. (8)) or two independent such eigenbivectors exist each with the same eigenvalue $\beta \neq 0$ ($\Rightarrow \alpha \neq 0$) in Eq. (8) and then the type is $\mathbf{D_1}$. The finding of general pnds amounts to solving Eq. (9(*ii*)) for E and is perhaps more conveniently done by writing this latter equation in the equivalent form $\overset{+}{W}_{abcd} E^{ab} E^{cd} = 0$ with E *not* an eigenbivector of $\overset{+}{W}$. This last equation results in a polynomial equation of order at most 4 for *real* solutions for E. Such solutions can then be calculated from Eq. (2)-Eq. (7). The resulting set of (real) solutions gives the complete set of solutions for principal $2-$spaces and pnds (repeated pnds arising if E is an eigenbivector of $\overset{+}{W}$ and general pnds otherwise) and these solutions can be shown to justify the term "repeated". It is remarked here that "real" solutions are required. This is because the general solutions of these polynomials sometimes contain complex totally null bivectors as solutions. The blades of such solutions actually contain no non-zero real vectors (up to scaling) and are thus rejected in this analysis (Hall, 2016).

Of course, similar results apply to $\overset{-}{W}$ and $\overset{-}{S}_m$ and the repeated and general pnds collectively give a description of C. To see this consider the

following equations for $C(m)$, for a non-zero $k \in T_m M$, for a $1-$form p at m which is *neither zero nor parallel to* k and with $\alpha \in \mathbb{R}$.

$$(i)\ C_{abcd}k^b k^d = \alpha k_a k_c \qquad (ii)\ C_{abcd}k^b k^d = k_a p_c + p_a k_c \tag{10}$$

If $\alpha \neq 0$ in (i), k is necessarily null but this is not true if $\alpha = 0$ (see (Hall, 2017; Hall, 2016)). So suppose that Eq. (10(i)) holds *with k assumed null*. Then k is said to span a *repeated principal null direction of* $C(m)$ (a repeated pnd). If Eq. (10(ii)) holds, k is necessarily null (and orthogonal to p) and is said to span a *general principal null direction of* $C(m)$ (a general pnd). [A set of equivalent statements to Eq. (10(ii)) are that (a) $k_{[e}C_{a]bc[d}k_{f]}k^b k^c = 0$ at m and (b) that Eq. (10(i)) is false]. Collectively, repeated and general pnds of C are referred to as *pnds* of C. Such directions are related to the analogous ones for $\overset{+}{W}$ and $\overset{-}{W}$ by the following lemma.

Lemma 2

A vector $k \in T_m M$ spans a repeated pnd for C if and only if it spans a repeated pnd for $\overset{+}{W}$ *and* $\overset{-}{W}$. A vector $k \in T_m M$ spans a general pnd for C if and only if it spans a pnd for $\overset{+}{W}$ *and* $\overset{-}{W}$ and is general for at least one of them.

It is noted and easily shown that any real eigenbivector of $C(m)$ is either a member of $\overset{+}{S}_m$ or $\overset{-}{S}_m$ or, if not, lies in an eigenspace of C spanned by eigenbivectors in $\overset{+}{S}_m$ or $\overset{-}{S}_m$. Thus one may think of all the eigenbivectors of C as being in $\overset{+}{S}_m$ or $\overset{-}{S}_m$. In fact, a canonical form for $C(m)$ is obtained from Eq. (1) by simply adding together canonical forms for $\overset{+}{W}(m)$ and $\overset{-}{W}(m)$ and the Segre type of $C(m)$ is simply the "sum" of the Segre types of $\overset{+}{W}(m)$ and $\overset{-}{W}(m)$ (with any brackets denoting degeneracies appropriately inserted). To determine the pnds of $C(m)$ one notes the following easily checked result that the intersection of two totally null $2-$spaces each of which lies in $\overset{+}{S}_m$ or each of which lies in $\overset{-}{S}_m$ is just the trivial subspace whereas the intersection of two totally null $2-$spaces one of which lies in $\overset{+}{S}_m$ and the other in $\overset{-}{S}_m$ is a null

direction at m. Thus when the principal $2-$spaces of $\overset{+}{W}(m)$ and $\overset{-}{W}(m)$ are known (and which lie, respectively, in $\overset{+}{S}_m$ and $\overset{-}{S}_m$) their intersections give the pnds of $C(m)$ according to lemma 2. The algebraic type of $C(m)$ can then be labelled $(\mathbf{A},\mathbf{B})$ where $\mathbf{A}$ and $\mathbf{B}$ are the algebraic types for $\overset{+}{W}(m)$ and $\overset{-}{W}(m)$. For example, $C(m)$ has type $(\mathbf{N},\mathbf{N})$ if and only if there exists a unique null direction spanned by k at m satisfying $C_{abcd}k^d = 0$ and which is the intersection of the (unique) repeated principal $2-$spaces for $\overset{+}{W}(m)$ and $\overset{-}{W}(m)$ for type $\mathbf{N}$. A consequence of this classification is the fact that there are finitely many (real) principal $2-$spaces for $\overset{+}{W}(m)$ and $\overset{-}{W}(m)$ (possibly none---see an earlier remark) and hence finitely many pnds for $C(m)$ (possibly none) except when the latter's algebraic type is of the form $(\mathbf{A},\mathbf{O})$ for certain choices of $\mathbf{A}$ (*e.g.*, type $(\mathbf{N},\mathbf{O})$) when infinitely many pnds occur.

Of course, the above classification is pointwise on M. However, one can display a topological decomposition of (an open dense subset of) M into *open* subsets of M on which the algebraic types of $\overset{+}{W}$, $\overset{-}{W}$ and C are constant. Also one can demonstrate the local smoothness (in an obvious sense) of the canonical forms and decompositions described in section 3 as well as study the isotropies arising from the the tetrad changes which preserve the given canonical forms for $\overset{+}{W}$, $\overset{-}{W}$ and C. This will be published elsewhere (Hall, 2017). In this last respect the study of the subalgebra structure of $o(2,2)$ given in (Ghanam and Thompson, 2001) and, in a more accessible form for the present purposes in (Wang and Hall, 2013), is useful.

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OBSERVAȚII ASUPRA CLASIFICĂRII TENSORULUI CONFORM WEYL ÎN VARIETĂȚI 4-DIMENSIONALE DE SIGNATURĂ NEUTRĂ

(Rezumat)

Această lucrare prezintă o scurtă discuție asupra clasificării algebrice a tensorului conform Weyl pe o varietate 4-dimensională cu metrică *g* de signatură neutră (+,+,-,-). Din punct de vedere algebric, clasificarea este similară cu binecunoscuta clasificare Petrov în cazul Lorentz. Sunt obținute diferite tipuri algebrice și formele canonice corespunzătoare. Sunt descrise mai multe detalii ale 2-spațiilor principale total nule și ale direcțiilor nule, similare celor ale lui L. Bel din cazul Lorentz.

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FIXED POINT THEOREMS FOR TWO PAIRS OF MAPPINGS SATISFYING COMMON LIMIT RANGE PROPERTY IN *G* – METRIC SPACES

BY

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Abstract. The purpose of this paper is to prove a general fixed point theorem for two pairs of mappings in *G* - metric spaces, generalizing the results from (Popa and Patriciu, 2014) and unifying the results from (Giniswamy and Maheshwari, 2014). Also, a new result for a sequence of mappings is obtained. In the last part of this paper as applications, some fixed point results for mappings satisfying contractive conditions of integral type, for almost contractive mappings, for ϕ - contractive mappings and (ϕ,ψ) - contractive mappings in *G* - metric spaces, are obtained.

Keywords: fixed point; almost altering distance; common limit range property; implicit relation; *G* - metric space.

1. Introduction

Let (X,d) be a metric space and S,T be two mappings of X . In 1996, Jungck (Jungck, 1996) defined S and T to be compatible if

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$$\lim_{n\to\infty} d(TSx_n, STx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t,$$

for some $t \in X$.

This concept has been frequently used to prove the existence theorems in fixed point theory.

Let f, g be self mappings of a nonempty set X. A point $x \in X$ is a coincidence point of f and g if $w = fx = gx$ and w is said to be a point of coincidence of f and g. The set of all coincidence points of f and g is denoted by $C(f, g)$.

In 1994, Pant (Pant, 1994) introduced the notion of pointwise R - weakly commuting mapping, which is equivalent to commutativity at coincidence points.

In 1996, Jungck (Jungck, 1996) introduced the notion of weakly compatible mappings.

Definition 1.1 (Jungck, 1996) *Let X be a nonempty set and f, g be self mappings of X. f and g are weakly compatible if $fgu = gfu$ for all $u \in C(f, g)$.*

Hence, f and g are weakly compatible if and only if f and g are pointwise R - weakly commuting.

The study of common fixed points for noncompatible mappings is also interesting, the work of this regard beeing initiated by Pant in (Pant, 1998; 1999).

Aamri and El - Moutawakil (2002) introduced a generalization of noncompatible mappings.

Definition 1.2 (Aamri and El - Moutawakil, 2002) *Let S and T be two self mappings of a metric space (X, d). We say that S and T satisfy property (EA) if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t,$$

for some $t \in X$.

Remark 1.1 *It is clear that two self mappings S and T of a metric space (X, d) will be noncompatible if there exists $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$ but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is non zero or non existent.*

Therefore, two noncompatible self mappings of a metric space (X,d) satisfy property (EA).

It is known from (Pathak *et al.*, 2010) that the notions of weakly compatible mappings and mappings satisfying property (EA) are independent.

There exists a vast literature concerning the study of fixed points for pairs of mappings satisfying property (EA).

In 2005, Liu *et al*. (Liu *et al.*, 2005) defined the notion of common property (EA).

Definition 1.3 (Liu *et al.*, 2005) *Two pairs* (A,S) *and* (B,T) *of self mappings of a metric space* (X,d) *are said to satisfy common property* (EA) *if there exist two sequences* $\{x_n\}$ *and* $\{y_n\}$ *in* X *such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t,$$

for some $t \in X$.

In 2011, Sintunavarat and Kumam (Sintunavarat and Kumam, 2011) introduced the notion of common limit range property.

Definition 1.4 (Sintunavarat and Kumam, 2011) *A pair* (A,S) *of self mappings of a metric space* (X,d) *is said to satisfy the common limit range property with respect to* S, *denoted* $CLR_{(S)}$ *if there exists a sequence* $\{x_n\}$ *in* X *such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t,$$

for some $t \in S(X)$.

Thus we can infer that a pair (A,S) satisfying the property (EA) along with the closedness of the subspace $S(X)$ always has the $CLR_{(S)}$ - property with respect to S (see Examples 2.16, 2.17 (Imdad *et al.*, 2012)).

Recently, Imdad *et al.* (2013) extended the notion of common limit range property to the pairs of self mappings.

Definition 1.5 (Imdad *et al.*, 2013) *Two pairs* (A,S) *and* (B,T) *of self mappings of a metric space* (X,d) *are said to satisfy common limit range property with respect to* S *and* T, *denoted* $CLR_{(S,T)}$ *if there exist two sequences* $\{x_n\}$ *and* $\{y_n\}$ *in* X *such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t,$$

where $t \in S(X) \cap T(X)$.

Some fixed point results for pairs of mappings with $CLR_{(S,T)}$ property are obtained in (Imdad and Chauhan, 2013; Karapinar *et al.*, 2013) and in other papers.

2. Preliminaries

In (Dhage, 1992; 2000), Dhage introduced a new class of generalized metric space, named D - metric spaces. Mustafa and Sims (2003; 2006), proved that most of the claims concerning the fundamental topological structures on D - metric spaces are incorrect and introduced appropriate notion of generalized metric space, named G - metric space. In fact, Mustafa, Sims and other authors studied many fixed point results for self mappings under certain conditions in (Mustafa *et al.*, 2008; Mustafa and Sims, 2009; Shatanawi, 2010), and in other papers.

Definition 2.1 (Mustafa and Sims, 2006) *Let X be a nonempty set and $G: X^3 \to \mathsf{R}_+$ be a function satisfying the following properties:*

$(G_1): G(x, y, z) = 0$ *for* $x = y = z$,

$(G_2): 0 < G(x, x, y)$ *for all* $x, y \in X$ *with* $x \neq y$,

$(G_3): G(x, y, y) \leq G(x, y, z)$ *for all* $x, y, z \in X$ *with* $z \neq y$,

$(G_4): G(x, y, z) = G(y, z, x) = G(z, x, y) = ...$ (*symmetry in all three variables*),

$(G_5): G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *for all* $x, y, z, a \in X$ (*triangle inequality*).

The function G is called a G - metric on X and (X, G) is called a G - metric space.

Note that if $G(x, y, z) = 0$, then $x = y = z$.

Remark 2.1 *Let (X, G) be a G - metric space. If $y = z$, then $G(x, y, y)$ is a quasi - metric on X. Hence, (X, Q), where $Q(x, y) = G(x, y, y)$, is a quasi - metric space and since every metric space is a particular case of quasi - metric space it follows that the notion of G - metric space is a generalization of a metric space.*

Definition 2.2 (Mustafa and Sims, 2006) *Let (X, G) be a G - metric space. A sequence $\{x_n\}$ in X is said to be:*

a) *G - convergent if for $\varepsilon > 0$, there exist $x \in X$ and $k \in \mathsf{N}$ such that for all $m, n \in \mathsf{N}, m, n \geq k$, $G(x, x_n, x_m) < \varepsilon$.*

b) *G - Cauchy if for $\varepsilon > 0$, there exists $k \in \mathsf{N}$ such that for all $m, n, p \in \mathsf{N}$, $m, n, p \geq k$, $G(x_n, x_m, x_p) < \varepsilon$, that is $G(x_n, x_m, x_p) \to 0$ as $n, m, p \to \infty$.*

c) *A G - metric space is said to be G - complete if every G - Cauchy sequence in X is G - convergent.*

Lemma 2.1 (Mustafa and Sims, 2006) *Let (X, G) be a G - metric space. Then, the following conditions are equivalent:*

1) *$\{x_n\}$ is G - convergent to x;*

2) *$G(x_n, x_n, x) \to 0$ as $n \to \infty$;*

3) $G(x_n, x, x) \to 0$ *as* $n \to \infty$ *;*

4) $G(x_n, x_m, x) \to 0$ *as* $n, m \to \infty$.

Lemma 2.2 (Mustafa and Sims, 2006) *If* (X, G) *is a* G *- metric space, then the following conditions are equivalent:*

1) $\{x_n\}$ *is* G *- Cauchy;*

2) *For* $\varepsilon > 0$ *, there exists* $k \in \mathbb{N}$ *such that* $G(x_n, x_m, x_m) < \varepsilon$ *for all* $m, n \in \mathbb{N}$, $m, n \geq k$.

Lemma 2.3 (Mustafa and Sims, 2006) *Let* (X, G) *be a* G *- metric space. Then, the function* $G(x, y, z)$ *is jointly continuous in all three of its variables.*

Definition 2.3 (Mustafa and Sims, 2006) *A* G *- metric on a set* X *is said to be symmetric if* $G(x, y, y) = G(y, x, x)$ *for all* $x, y \in X$ *. Then,* (X, G) *is said to be symmetric* G *- metric space.*

Quite recently (Popa and Patriciu, 2014), a general fixed point theorem for a pair of mappings satisfying $CLR_{(S)}$ - property in G - metric spaces is proved.

Definition 2.4 (Khan *et al.*, 1984) *An altering distance is a function* $\phi : [0, \infty) \to 0, \infty)$ *satisfying:*

(ϕ_1): ϕ *is increasing and continuous;*

(ϕ_2): $\phi(t) = 0$ *if and only if* $t = 0$.

Fixed point theorems involving altering distances have been studied in (Popa and Mocanu, 2007; Sastri and Babu, 1998; 1999) and in other papers.

Definition 2.5 (Popa and Patriciu, 2014) *A function* $\psi : [0, \infty) \to 0, \infty)$ *is an almost altering distance if:*

(ψ_1): ψ *is continuous;*

(ψ_2): $\psi(t) = 0$ *if and only if* $t = 0$.

Remark 2.1 *Every altering distance is an almost altering distance, but the converse is not true.*

Example 2.1 $\psi(t) = \begin{cases} t, t \in [0,1] \\ \dfrac{1}{t}, t \in (1, \infty). \end{cases}$

3. Implicit Relations in G - Metric Spaces

Several fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit function in (Popa, 1997; 1999) and in other papers.

Recently, the method is used in the study of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, b - metric spaces, ultra - metric spaces, reflexive spaces, compact metric spaces, paracompact metric spaces, in two and three metric spaces, for single - valued mappings, hybrid pairs of mappings and set - valued mappings. The method is used in the study of fixed points for mappings satisfying a contractive/extensive condition of integral type, in fuzzy metric spaces, probabilistic metric spaces, intuitionistic metric spaces, partial metric spaces and G - metric spaces.

The study of fixed points for mappings satisfying implicit relations in G - metric spaces is initiated in (Popa and Patriciu, 2012; 2013) and in other papers.

With this method the proofs of some fixed point theorems are more simple. Also, the method allows the study of local and global properties of fixed point structures.

The study of fixed points for pairs of self mappings with common limit range property in metric spaces satisfying implicit relations is initiated in (Imdad and Chauhan, 2013).

The study of fixed points for a pair of self mappings with common limit range property in G - metric spaces is initiated in (Popa and Patriciu, 2014).

In 2008, Ali and Imdad (Ali and Imdad, 2008) introduced a new class of implicit relations.

Definition 3.1 (Ali and Imdad, 2008) *Let* F_G *be the family of lower semi - continuous functions* $F:\mathsf{R}_+^6 \to \mathsf{R}$ *satisfying the following conditions:*

(F_1): $F(t,0,t,0,0,t) > 0$, *for all* $t > 0$;

(F_2): $F(t,0,0,t,t,0) > 0$, *for all* $t > 0$;

(F_3): $F(t,t,0,0,t,t) > 0$, *for all* $t > 0$.

Example 3.1 $F(t_1,...,t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, *where* $a,b,c,d,e \geq 0$ *and* $a+b+c+d+e < 1$.

Example 3.2 $F(t_1,...,t_6) = t_1 - k\max\left\{t_2,t_3,t_4,\dfrac{t_5+t_6}{2}\right\}$, *where* $k \in [0,1)$.

Example 3.3 $F(t_1,...,t_6) = t_1 - k\max\{t_2,t_3,...,t_6\}$, *where* $k \in [0,1)$.

Example 3.4 $F(t_1,...,t_6) = t_1 - k\max\left\{t_2,\dfrac{t_3+t_4}{2},\dfrac{t_5+t_6}{2}\right\}$, *where* $k \in [0,1)$.

Example 3.5 $F(t_1,...,t_6) = t_1 - at_2 - b\max\{t_3,t_4\} - c\max\{t_2,t_5,t_6\}$, *where* $a,b,c \geq 0$ *and* $a+b+c < 1$.

Example 3.6 $F(t_1,...,t_6) = t_1 - \alpha\max\{t_2,t_3,t_4\} - (1-\alpha)(at_5 + bt_6)$, *where* $\alpha \in (0,1)$, $a,b \geq 0$ *and* $a+b < 1$.

Example 3.7 $F(t_1,...,t_6) = t_1 - at_2 - b(t_3 + t_4) - c\min\{t_5, t_6\}$, *where* $a,b,c > 0$ *and* $a + b + c < 1$.

Example 3.8 $F(t_1,...,t_6) = t_1 - at_2 - \dfrac{b(t_5 + t_6)}{1 + t_3 + t_4}$, *where* $a,b \geq 0$ *and* $a + 2b < 1$.

Example 3.9 $F(t_1,...,t_6) = t_1 - \max\{ct_2, ct_3, ct_4, at_5 + bt_6\}$, *where* $c \in (0,1)$, $a,b \geq 0$ *and* $a + b + c < 1$.

Quite recently, the following theorem is proved in (Popa & Patriciu, 2014).

Theorem 3.1 (Popa & Patriciu, 2014) *Let* T *and* S *be self mappings of a* G *- metric space* (X,G) *such that*

$$F(\psi(G(Tx,Tx,Ty)),\psi(G(Sx,Sx,Sy)),\psi(G(Tx,Tx,Sx)),$$
$$\psi(G(Ty,Ty,Sy)),\psi(G(Sx,Sx,Ty)),\psi(G(Tx,Sx,Sy))) < 0,$$

for all $x,y \in X$, *where* F *satisfies properties* $(F_1),(F_3)$ *and* ψ *is an almost altering distance. If* T *and* S *satisfy* $CLR_{(S)}$ *- property, then* $C(T,S) \neq \varnothing$. *Moreover, if* T *and* S *are weakly compatible, then* T *and* S *have a unique common fixed point.*

The purpose of this paper is to prove a general fixed point theorem for two pairs of mappings satisfying common limit range property in G - metric spaces, generalizing the results from (Popa and Patriciu, 2014) and unifying the results from (Giniswamy and Maheshwari, 2014). Also, a new result for a sequence of mappings is obtained.

In the last part of this paper, as applications, some fixed point results for mappings satisfying contractive conditions of integral type, for almost contractive mappings, for φ - contractive mappings and (φ,ψ) - contractive mappings in G - metric spaces are obtained.

4. Main Results

Lemma 4.1 (Abbas and Rhoades, 2009) *Let* f,g *be two weakly compatible self mappings of a nonempty set* X. *If* f *and* g *have a unique point of coincidence* $w = fx = gx$ *for some* $x \in X$, *then* w *is the unique common fixed point of* f *and* g.

Theorem 4.1 *Let* A,B,S *and* T *be self mappings of a* G *- metric space* (X,G) *satisfying inequality*

$$F(\psi(G(Ax,By,By)),\psi(G(Sx,Ty,Ty)),\psi(G(Sx,Sx,Ax)),$$
$$\psi(G(Ty,By,By)),\psi(G(Sx,By,By)),\psi(G(Ax,Ty,Ty))) \leq 0, \quad (4.1)$$

for all $x, y \in X$, F *satisfies property* (F_3) *and* ψ *is an almost altering distance.*

If there exist $u, v \in X$ *such that* $Au = Su$ *and* $Bv = Tv$, *then there exists* $t \in X$ *such that* t *is the unique point of coincidence of* A *and* S , *as well* t *is the unique point of coincidence of* B *and* T .

Proof. First we prove that $Su = Tv$. Suppose that $Su \neq Tv$. By (4.1) we obtain

$$F(\psi(G(Au, Bv, Bv)), \psi(G(Su, Tv, Tv)), \psi(G(Su, Su, Au)),$$
$$\psi(G(Tv, Bv, Bv)), \psi(G(Su, Bv, Bv)), \psi(G(Au, Tv, Tv))) \leq 0,$$

$$F(\psi(G(Su, Tv, Tv)), \psi(G(Su, Tv, Tv)), 0, 0, \psi(G(Su, Tv, Tv)), \psi(G(Su, Tv, Tv))) \leq 0,$$

a contradiction of (F_3).

Hence, $Su = Tv$, which implies $Su = Au = Bv = Tv = t$. Suppose that there exists $z = Aw = Sw$ with $z \neq t$. Then, by (4.1) we obtain

$$F(\psi(G(Aw, Bv, Bv)), \psi(G(Sw, Tv, Tv)), \psi(G(Sw, Sw, Aw)),$$
$$\psi(G(Tv, Bv, Bv)), \psi(G(Sw, Bv, Bv)), \psi(G(Aw, Tv, Tv))) \leq 0,$$

$$F(\psi(G(Sw, Tv, Tv)), \psi(G(Sw, Tv, Tv)), 0, 0, \psi(G(Sw, Tv, Tv)), \psi(G(Sw, Tv, Tv))) \leq 0,$$

a contradiction of (F_3).

Hence, $z = Sw = Aw = Tv = Bv = Au = Su = t$ and t is the unique point of coincidence of A and S. Similarly, t is the unique point of coincidence of B and T.

Theorem 4.2 *Let* A, B, S *and* T *be self mappings of a* G *- metric space* (X, G) *satisfying inequality (4.1) for all* $x, y \in X$, $F \in \mathsf{F}_G$ *and* ψ *is an almost altering distance. If* (A, S) *and* (B, T) *satisfy* $CLR_{(S,T)}$ *- property, then*

i) $C(A, S) \neq \emptyset$,

ii) $C(B, T) \neq \emptyset$.

Moreover, if (A, S) *and* (B, T) *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point.*

Proof. Since (A, S) and (B, T) satisfy $CLR_{(S,T)}$ - property, there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

where $z \in S(X) \cap T(X)$.

Since $z \in T(X)$, there exists $u \in X$ such that $z = Tu$.

By (4.1) we have

$$F(\psi(G(Ax_n, Bu, Bu)), \psi(G(Sx_n, Tu, Tu)), \psi(G(Sx_n, Sx_n, Ax_n)),$$
$$\psi(G(Tu, Bu, Bu)), \psi(G(Sx_n, Bu, Bu)), \psi(G(Ax_n, Tu, Tu))) \leq 0.$$

Letting n tends to infinity we obtain

$$F(\psi(G(z,Bu,Bu)),0,0,\psi(G(z,Bu,Bu)),\psi(G(z,Bu,Bu)),0)\le 0,$$

a contradiction of (F_2) if $\psi(G(z,Bu,Bu))>0$. Hence, $\psi(G(z,Bu,Bu))=0$, which implies $z=Bu=Tu$ and $C(B,T)\neq\varnothing$.

Since $z\in S(X)$, there exists $v\in X$ such that $z=Sv$. By (4.1) we obtain

$$F(\psi(G(Av,Bu,Bu)),\psi(G(Sv,Tu,Tu)),\psi(G(Sv,Sv,Av)),$$
$$\psi(G(Tu,Bu,Bu)),\psi(G(Sv,Bu,Bu)),\psi(G(Av,Tu,Tu)))\le 0,$$

$$F(\psi(G(Av,z,z)),0,\psi(G(Av,z,z)),0,0,\psi(G(Av,z,z)))\le 0,$$

a contradiction of (F_1) if $\psi(G(Av,z,z))>0$. Hence, $\psi(G(Av,z,z))=0$, which implies $z=Av=Sv$ and $C(A,S)\neq\varnothing$.

By Theorem 4.1, z is the unique point of coincidence of (A,S) and (B,T).

Moreover, if (A,S) and (B,T) are weakly compatible, by Lemma 4.1, z is the unique fixed point of A,B,S and T.

If $\psi(t)=t$, then by Theorem 4.2 we obtain

Theorem 4.3 *Let A,B,S and T be self mappings of a G - metric space (X,G) satisfying the inequality*

$$\begin{aligned}&F(G(Ax,By,By),G(Sx,Ty,Ty),G(Sx,Sx,Ax),\\&G(Ty,By,By),G(Sx,By,By),G(Ax,Ty,Ty))\le 0,\end{aligned}\tag{4.2}$$

for all $x,y\in X$, $F\in \mathsf{F}_G$.

If (A,S) and (B,T) satisfy $CLR_{(S,T)}$ - property, then

i) $C(A,S)\neq\varnothing$,

ii) $C(B,T)\neq\varnothing$.

Moreover, if (A,S) and (B,T) are weakly compatible, then A,B,S and T have a unique common fixed point.

Example 4.1 Let $X=[0,11]$ and let $G:X^3\to\mathbb{R}_+$ be the $G-$ metric defined as follows

$$G(x,y,z)=\max\{|x-y|,|y-z|,|x-z|\}$$

for all $x,y,z\in X$. Then (X,G) is a $G-$metric space.

Define the self mappings A,B,S and T

$$Ax=\begin{cases}2, x\in[0,2]\cup(5,11]\\ 5, x\in(2,5],\end{cases} \qquad Sx=\begin{cases}2, x\in[0,2]\\ 6, x\in(2,5)\\ \dfrac{3x+1}{8}, x\in[5,11],\end{cases}$$

$$Bx=\begin{cases}2, x\in[0,2]\cup(5,11]\\ 4, x\in(2,5],\end{cases} \qquad Tx=\begin{cases}2, x\in[0,2]\\ 8, x\in(2,5]\\ x-3, x\in(5,11].\end{cases}$$

Then

$$AX=\{2,5\}, BX=\{2,4\}, SX=\left[2,\frac{17}{4}\right)\cup\{6\}, TX=[2,8].$$

Let $x_n=2-\dfrac{1}{n}$ and $y_n=2-\dfrac{1}{n^2}$ be. Then

$$\lim Ax_n=\lim Sx_n=\lim By_n=\lim Ty_n=2\in S(X)\cap T(X)$$

and (A,S) and (B,T) satisfies $CLR_{(S,T)}$ − property.

On the other hand, $z=2$ is the unique point of coincidence of (A,S) and (B,T).

$Ax=Sx$ for $x\in[0,2]$, $Bx=Tx$ for $x\in[0,2]$, $ASx=SAx=2$. Similarly, $BTx=TBx=2$, hence (A,S) and (B,T) are weakly compatible.

If

$$M(x,y)=\max\{G(Sx,Ty,Ty), G(Sx,Sx,Ax),$$
$$G(Ty,By,By), G(Sx,By,By), G(Ax,Ty,Ty)\},$$

then by a routine calculation we obtain

$$G(Ax,By,By)\le kM(x,y),$$

with $k\in\left[\dfrac{3}{4},1\right)$.

Thus, by Example 1 and Theorem 4.2, A, B, S and T have a unique common fixed point which is $x=2$.

Similarly as in Theorem 4.2 we obtain

Theorem 4.4 *Let A, B, S and T be self mappings of a G - metric space (X,G) satisfying inequality*

$$\begin{aligned}&F(\psi(G(Ax,Ax,By)),\psi(G(Sx,Sx,Ty)),\psi(G(Sx,Ax,Ax)),\\ &\psi(G(Ty,Ty,By)),\psi(G(Sx,Sx,By)),\psi(G(Ax,Ax,Ty)))\le 0,\end{aligned} \tag{4.3}$$

for all $x,y\in X$, $F\in\mathsf{F}_G$ and ψ is an almost altering distance.

If (A,S) *and* (B,T) *satisfy* $CLR_{(S,T)}$ *- property, then*

i) $C(A,S) \neq \emptyset$,

ii) $C(B,T) \neq \emptyset$.

Moreover, if (A,S) *and* (B,T) *are weakly compatible, then* A,B,S *and* T *have a unique common fixed point.*

Theorem 4.5 *Let* (X,G) *be a* G *- metric space and* A,B,S *and* T *be self mappings of* X *satisfying the inequality*

$$\begin{aligned} G(Ax,By,By) \leq aG(Sx,Ty,Ty) + bG(Sx,Sx,Ax) + \\ + cG(Ty,By,By) + dG(Sx,By,By) + eG(Ax,Ty,Ty), \end{aligned} \tag{4.4}$$

for all $x,y \in X$, $a,b,c,d,e \geq 0$ *and* $a+b+c+d+e<1$.

If (A,S) *and* (B,T) *satisfy* $CLR_{(S,T)}$ *- property, then*

i) $C(A,S) \neq \emptyset$,

ii) $C(B,T) \neq \emptyset$.

Moreover, if (A,S) *and* (B,T) *are weakly compatible, then* A,B,S *and* T *have a unique common fixed point.*

Corollary 4.1 (**Theorem 2.5** (Giniswamy and Maheshwari, 2014)) *Let* (X,G) *be a* G *- metric space and* A,B,S *and* T *be self mappings of* X *such that:*

1) (A,S) and (B,T) satisfy $CLR_{(S,T)}$ - property;

2)
$$\begin{aligned} G(Ax,By,Bz) \leq pG(Sx,Ty,Ty) + qG(Sx,Sx,Ax) + \\ + rG(Ty,Bz,Bz) + t[G(Ax,Ty,Tz) + G(Sx,By,Bz)], \end{aligned} \tag{4.5}$$

for all $x,y,z \in X$, *where* $p,q,r,t \geq 0$ *and* $p+q+r+2t<1$.

Then (A,S) *and* (B,T) *have a unique point of coincidence in* X .

Moreover, if (A,S) *and* (B,T) *are weakly compatible, then* A,B,S *and* T *have a unique common fixed point.*

Proof. Let $y = z$, then by (4.5) we obtain a particular case of (4.4) and the proof follows from Theorem 4.5.

Theorem 4.6 *Let* (X,G) *be a* G *- metric space and* A,B,S *and* T *be self mappings of* X *satisfying the inequality:*

$$\begin{aligned} G(Ax,By,By) \leq k \max\{G(Sx,Ty,Ty), G(Sx,Sx,Ax), \\ G(Ty,By,By), \frac{G(Sx,By,By)+G(Ax,Ty,Ty)}{2}\}, \end{aligned} \tag{4.6}$$

for all $x,y \in X$ *and* $k \in [0,1)$.

If (A,S) *and* (B,T) *satisfy* $CLR_{(S,T)}$ *- property, then*

i) $C(A,S) \neq \emptyset$,

ii) $C(B,T) \neq \emptyset$.

Moreover, if (A,S) *and* (B,T) *are weakly compatible, then* A,B,S *and* T *have a unique common fixed point.*

Proof. The proof follows from Theorem 4.3 and Example 3.2.

Corollary 4.2 (**Theorem 2.6** (Giniswamy and Maheshwari, 2014)) *Let* (X,G) *be a* G *- metric space and* A,B,S *and* T *be self mappings of* X *such that:*

1) (A,S) *and* (B,T) *satisfy* $CLR_{(S,T)}$ *- property;*

2) $G(Ax,By,Bz) \le hu(x,y,z)$, *where* $h \in (0,1)$, $x,y,z \in X$ *and*

$$u(x,y,z) \in \{G(Ax,Sx,Sx), G(Sx,Ty,Ty), G(Ty,By,By), \frac{G(Ax,Ty,Tz)+G(Sx,By,Bz)}{2}\}.$$

Then (A,S) *and* (B,T) *have a unique point of coincidence in* X.

Moreover, if (A,S) *and* (B,T) *are weakly compatible, then* A,B,S *and* T *have a unique common fixed point.*

Proof. Let $y = z$, then by (2) we obtain

$$G(Ax,By,By) \le h\max\{G(Sx,Ty,Ty), G(Sx,Sx,Ax),$$
$$G(Ty,By,By), \frac{G(Sx,By,By)+G(Ax,Ty,Ty)}{2}\},$$

which is inequality (4.6) and the proof of Corollary 4.2 follows from Theorem 4.6.

For a function $f : X \to X$ we denote

$$Fix(f) = \{x \in X : x = fx\}.$$

Theorem 4.7 *Let* A,B,S *and* T *be self mappings of a* G *- metric space* (X,G). *If the inequality (4.1) holds for all* $x,y \in X$, $F \in \mathsf{F}_G$ *and* ψ *is an almost altering distance, then*

$$[Fix(S) \cap Fix(T)] \cap Fix(A) = [Fix(S) \cap Fix(T)] \cap Fix(B).$$

Proof. Let $x \in [Fix(S) \cap Fix(T)] \cap Fix(A)$. Then by (4.1) we have

$$F(\psi(G(Ax,Bx,Bx)), \psi(G(Sx,Tx,Tx)), \psi(G(Sx,Sx,Ax)),$$
$$\psi(G(Tx,Bx,Bx)), \psi(G(Sx,Bx,Bx)), \psi(G(Ax,Tx,Tx))) \le 0,$$
$$F(\psi(G(x,Bx,Bx)),0,0,\psi(G(x,Bx,Bx)),\psi(G(x,Bx,Bx)),0) \le 0,$$

a contradiction of (F_2) if $\psi(G(x,Bx,Bx)) > 0$. Hence, $\psi(G(x,Bx,Bx)) = 0$ which implies $x = Bx$ and $x \in Fix(B)$.

Hence

$$[Fix(S) \cap Fix(T)] \cap Fix(A) \subset [Fix(S) \cap Fix(T)] \cap Fix(B).$$

Similarly, by (4.1) and (F_1) we obtain

$$[Fix(S) \cap Fix(T)] \cap Fix(B) \subset [Fix(S) \cap Fix(T)] \cap Fix(A).$$

Theorems 4.2 and 4.7 imply the following one.

Theorem 4.8 *Let* S,T *and* $\{A_i\}_{i\in\mathbb{N}^*}$ *be self mappings of a* G *- metric space* (X,G) *satisfying the inequality*

$$\begin{aligned} &F(\psi(G(A_i x, A_{i+1}y, A_{i+1}y)),\psi(G(Sx,Ty,Ty)),\psi(G(Sx,Sx,A_i x)),\\ &\psi(G(Ty,A_{i+1}y,A_{i+1}y)),\psi(G(Sx,A_{i+1}y,A_{i+1}y)),\psi(G(A_i x,Ty,Ty)))\le 0, \end{aligned} \tag{4.7}$$

for all $x,y\in X$, $F\in\mathsf{F}_G$, ψ *is an almost altering distance and* $i\in\mathbb{N}^*$.

If (A_1,S) *and* (A_2,T) *satisfy* $CLR_{(S,T)}$ *- property and* $(A_1,S),(A_2,T)$ *are weakly compatible, then* S,T *and* $\{A_i\}_{i\in\mathbb{N}^*}$ *have a unique common fixed point.*

If $\psi(t)=t$, from Theorem 4.8 we obtain

Theorem 4.9 *Let* S,T *and* $\{A_i\}_{i\in\mathbb{N}^*}$ *be self mappings of a* G *- metric space* (X,G) *satisfying the inequality*

$$\begin{aligned} &F(G(A_i x,A_{i+1}y,A_{i+1}y),G(Sx,Ty,Ty),G(Sx,Sx,A_i x),\\ &G(Ty,A_{i+1}y,A_{i+1}y),G(Sx,A_{i+1}y,A_{i+1}y),G(A_i x,Ty,Ty))\le 0, \end{aligned} \tag{4.8}$$

for all $x,y\in X$, $F\in\mathsf{F}_G$ *and* $i\in\mathbb{N}^*$.

If (A_1,S) *and* (A_2,T) *satisfy* $CLR_{(S,T)}$ *- property and* $(A_1,S),(A_2,T)$ *are weakly compatible, then* S,T *and* $\{A_i\}_{i\in\mathbb{N}^*}$ *have a unique common fixed point.*

Remark 4.1 *We obtain similar results from Theorem 4.4.*

5. Applications

5.1. Fixed Points for Mappings Satisfying Contractive Conditions of Integral Type

In (Branciari, 2002), Branciari established the following theorem which opened the way to the study of fixed points for mappings satisfying contractive conditions of integral type.

Theorem 5.1 (Branciari, 2002) *Let* (X,d) *be a complete metric space,* $c\in(0,1)$ *and* $f:X\to X$ *such that for all* $x,y\in X$

$$\int_0^{d(fx,fy)} h(t)dt \le c\int_0^{d(x,y)} h(t)dt,$$

whenever $h:[0,\infty)\to[0,\infty)$ *is a Lebesgue measurable mapping which is summable (i.e., with finite integral) on each compact subset of* $[0,\infty)$ *such that*

$\int_0^{\varepsilon} h(t)dt > 0$ *for each* $\varepsilon > 0$. *Then,* f *has an unique fixed point* $z \in X$ *such that for all* $x \in X$, $z = \lim_{n\to\infty} f^n x$.

Theorem 5.1 has been extended to a pair of compatible mappings in (Kumar *et al.*, 2007).

Theorem 5.2 (Kumar *et al.*, 2007) *Let* f, g *be compatible mappings of a complete metric space with* g *– continuous satisfying the following conditions:*

1) $f(X) \subset g(X)$,

2) $\int_0^{d(fx,gy)} h(t)dt \le c \int_0^{d(x,y)} h(t)dt$,

for some $c \in (0,1)$, *whenever* $x, y \in X$ *and* $h(t)$ *as in Theorem 5.1.*

Then, f *and* g *have a unique common fixed point.*

Some fixed point results for mappings satisfying contractive conditions of integral type are proved in (Popa and Mocanu, 2007; 2009) and in other papers.

Lemma 5.1 Let $h:[0,\infty) \to 0,\infty)$ *as in Theorem 5.1. Then* $\psi(t) = \int_0^t h(x)dx$ *is an almost altering distance.*

Proof. The proof follows from Lemma 2.5 (Popa and Mocanu, 2009).

Theorem 5.3 *Let* A, B, S *and* T *be self mappings of a* G *- metric space* (X, G) *such that*

$$F\Big(\int_0^{G(Ax,By,By)} h(t)dt, \int_0^{G(Sx,Ty,Ty)} h(t)dt, \int_0^{G(Sx,Sx,Ax)} h(t)dt, \int_0^{G(Ty,By,By)} h(t)dt, \int_0^{G(Sx,By,By)} h(t)dt, \int_0^{G(Ax,Ty,Ty)} h(t)dt\Big) \le 0, \tag{5.1}$$

for all $x, y \in X$, *where* $F \in \mathsf{F}_G$ *and* $h(t)$ *as in Theorem 5.1.*

If (A,S) *and* (B,T) *satisfy* $CLR_{(S,T)}$ *- property, then*

i) $C(A,S) \neq \emptyset$,

ii) $C(B,T) \neq \emptyset$.

Moreover, if (A,S) *and* (B,T) *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point.*

Proof. By Lemma 5.1, $\psi(t) = \int_0^t h(x)dx$ is an almost altering distance. By (5.1) we have

$$F(\psi(G(Ax,By,By)), \psi(G(Sx,Ty,Ty)), \psi(G(Sx,Sx,Ax)), \psi(G(Ty,By,By)), \psi(G(Sx,By,By)), \psi(G(Ax,Ty,Ty))) \le 0.$$

Hence the conditions of Theorem 4.2 are satisfied and the conclusions of Theorem 5.3 follows.

Similarly, from Theorem 4.4 we obtain

Theorem 5.4 *Let A, B, S and T be self mappings of a G - metric space (X, G) such that*

$$F(\int_0^{G(Ax,Ax,By)} h(t)dt, \int_0^{G(Sx,Sx,By)} h(t)dt, \int_0^{G(Sx,Sx,Ax)} h(t)dt, \int_0^{G(Ty,Ty,By)} h(t)dt, \int_0^{G(Sx,Sx,By)} h(t)dt, \int_0^{G(Ax,Ax,Ty)} h(t)dt) \le 0, \tag{5.2}$$

for all $x, y \in X$, where $F \in \mathsf{F}_G$ and $h(t)$ as in Theorem 5.1.

If (A, S) and (B, T) satisfy $CLR_{(S,T)}$ - property, then

i) $C(A, S) \neq \emptyset$,

ii) $C(B, T) \neq \emptyset$.

Moreover, if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point.

From Theorem 5.4 and Example 3.2 we obtain

Theorem 5.5 *Let (X, G) be a G - metric space and A, B, S and T be self mappings of X satisfying*

$$\int_0^{G(Ax,Ax,By)} h(t)dt \le k \max\{\int_0^{G(Sx,Sx,Ty)} h(t)dt, \int_0^{G(Sx,Sx,Ax)} h(t)dt, \int_0^{G(Ty,Ty,By)} h(t)dt, \frac{\int_0^{G(Sx,Sx,By)} h(t)dt + \int_0^{G(Ax,Ax,Ty)} h(t)dt}{2}\},$$

for all $x, y \in X$, $k \in [0,1)$ and $h(t)$ as in Theorem 5.1.

If (A, S) and (B, T) satisfy $CLR_{(S,T)}$ - property, then

i) $C(A, S) \neq \emptyset$,

ii) $C(B, T) \neq \emptyset$.

Moreover, if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point.

Remark 5.1 *If $h(t) = 1$, from Theorem 5.5 we obtain Theorem 4.6.*

From Theorems 5.3, 5.4 and Examples 3.1 – 3.9 we obtain new particular results.

5.2. Fixed Points for Almost Contractive Mappings in G - Metric Spaces

Definition 5.1 *Let (X, d) be a metric space. A mapping $T : X \to X$ is called weak contractive* (Berinde, 2003; 2004) *or almost contractive* (Berinde, 2010) *if there exist $\delta \in (0,1)$ and some $L \ge 0$ such that*

$$d(Tx, Ty) \le \delta d(x, y) + Ld(y, Tx) \text{ for all } x, y \in X.$$

The following theorem is proved in (Berinde, 2010).

Theorem 5.6 (Berinde, 2010) *Let (X,d) be a metric space and $T,S:X\to X$ be mappings for which there exists $a\in(0,1)$ and some $L\geq 0$ such that*

$$d(Tx,Ty)\leq ad(Sx,Sy)+Ld(Sy,Tx),$$

for all $x,y\in X$.

If $T(X)\subset S(X)$ and $S(X)$ is a complete subspace of X, then T and S have a unique point of coincidence. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point.

A similar result is obtained if

$$d(Tx,Ty)\leq ad(Sx,Sy)+L\min\{d(Sx,Tx),d(Sy,Ty),d(Sx,Ty),d(Tx,Sy)\},$$

where $a\in(0,1)$ and $L\geq 0$.

In (Babu *et el.*, 2008), a similar result is obtained if

$$d(Tx,Ty)\leq \delta m(x,y)+L\min\{d(Sx,Tx),d(Sy,Ty),d(Sx,Ty),d(Tx,Sy)\},$$

where $\delta\in(0,1)$, $L\geq 0$ and

$$m(x,y)=\max\{d(Sx,Sy),\frac{d(Tx,Sx)+d(Ty,Sy)}{2},\frac{d(Sx,Ty)+d(Tx,Sy)}{2}\}.$$

The following functions $F:\mathsf{R}_+^6\to\mathsf{R}$ satisfy conditions (F_1), (F_2) and (F_3).

Example 5.1 $F(t_1,...,t_6)=t_1-\delta\max\left\{t_2,\frac{t_3+t_4}{2},\frac{t_5+t_6}{2}\right\}-$

$-L\min\{t_3,t_4,t_5,t_6\}$, *where* $\delta\in(0,1)$ *and* $L\geq 0$.

Example 5.2 $F(t_1,...,t_6)=t_1-at_2-L\min\{t_3,t_4,t_5,t_6\}$, *where* $a\in(0,1)$ *and* $L\geq 0$.

Example 5.3 $F(t_1,...,t_6)=t_1-k\max\left\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\right\}-$

$-L\min\{t_3,t_4,t_5,t_6\}$, *where* $k\in(0,1)$ *and* $L\geq 0$.

Example 5.4 $F(t_1,...,t_6)=t_1-k\max\{t_2,t_3,t_4,t_5,t_6\}-$

$-L\min\{t_3,t_4,t_5,t_6\}$, *where* $k\in(0,1)$ *and* $L\geq 0$.

Example 5.5 $F(t_1,...,t_6)=t_1-k\max\left\{t_2,\frac{t_3+t_4}{2},\frac{t_5+t_6}{2}\right\}-$

$L\min\{t_3,t_4,\sqrt{t_4t_5},\sqrt{t_5t_6}\}$, *where* $k\in(0,1)$ *and* $L\geq 0$.

Example 5.6 $F(t_1,...,t_6)=t_1-k\max\{t_2,t_3,\sqrt{t_4t_5},\sqrt{t_5t_6}\}-$

$L\min\{t_3,t_4,t_5,t_6\}$, *where* $k\in(0,1)$ *and* $L\geq 0$.

Example 5.7 $F(t_1,...,t_6)=t_1-\max\{k(t_3+t_4),k(t_5+t_6)\}-$

$L\min\{t_3,t_4,t_5,t_6\}$, *where* $k\in\left(0,\frac{1}{2}\right)$ *and* $L\geq 0$.

Example 5.8 $F(t_1,...,t_6)=t_1-\max\left\{t_2,\alpha t_3,\alpha t_4,\frac{\alpha(t_5+t_6)}{2}\right\}-$ $L\min\{t_3,t_4,t_5,t_6\}$, *where* $\alpha\in(0,1)$ *and* $L\geq 0$.

By Theorem 4.2 and Example 5.1 we obtain

Theorem 5.7 *Let* A,B,S *and* T *be self mappings of a* G *- metric space* (X,G) *such that*

$$\psi(G(Ax,By,By))\leq\delta\max\{\psi(G(Sx,Ty,Ty)),$$
$$\frac{\psi(G(Sx,Sx,Ax))+\psi(G(Ty,By,By))}{2},\frac{\psi(G(Sx,By,By))+\psi(G(Ax,Ty,Ty))}{2}\},$$

where $\delta\in(0,1)$, $L\geq 0$, *for all* $x,y\in X$ *and* ψ *is an almost altering distance.*

If (A,S) *and* (B,T) *satisfy* $CLR_{(S,T)}$ *- property, then*

i) $C(A,S)\neq\emptyset$,

ii) $C(B,T)\neq\emptyset$.

Moreover, if (A,S) *and* (B,T) *are weakly compatible, then* A,B,S *and* T *have a unique common fixed point.*

Theorem 5.8 *Let* A,B,S *and* T *be self mappings of a* G *- metric space* (X,G) *such that*

$$\int_0^{G(Ax,By,By)}h(t)dt\leq\delta\max\{\int_0^{G(Sx,Ty,Ty)}h(t)dt,$$
$$\frac{\int_0^{G(Sx,Sx,Ax)}h(t)dt+\int_0^{G(Ty,By,By)}h(t)dt}{2},\frac{\int_0^{G(Sx,By,By)}h(t)dt+\int_0^{G(Ax,Ty,Ty)}h(t)dt}{2}\}+$$
$$L\min\{\int_0^{G(Sx,Sx,Ax)}h(t)dt,\int_0^{G(Ty,By,By)}h(t)dt,\int_0^{G(Sx,By,By)}h(t)dt,\int_0^{G(Ax,Ty,Ty)}h(t)dt\},$$

where $\delta\in(0,1)$ *and* $L\geq 0$, *for all* $x,y\in X$ *and* $h(t)$ *as in Theorem 5.1.*

If (A,S) *and* (B,T) *satisfy* $CLR_{(S,T)}$ *- property, then*

i) $C(A,S)\neq\emptyset$,

ii) $C(B,T)\neq\emptyset$.

Moreover, if (A,S) *and* (B,T) *are weakly compatible, then* A,B,S *and* T *have a unique common fixed point.*

Remark 5.2 *Similar results are obtained by Examples 5.2 – 5.8.*

5.3. Fixed Points for Mappings Satisfying φ - Contractive Conditions in G - Metric Spaces

As in (Matkowski, 1997), let ϕ be the set of all real nondecreasing continuous functions $\varphi:[0,\infty)\to 0,\infty)$ with $\lim_{n\to\infty}\varphi^n(t)=0$.

If $\varphi\in\phi$, then

1) $\varphi(t)<t$ for all $t\in(0,\infty)$,

2) $\varphi(0)=0$.

The following functions $F:\mathsf{R}_+^6\to\mathsf{R}$ satisfy conditions (F_1), (F_2) and (F_3).

Example 5.9 $F(t_1,...,t_6)=t_1-\varphi(\max\{t_2,t_3,t_4,t_5,t_6\})$.

Example 5.10 $F(t_1,...,t_6)=t_1-\varphi\left(\max\left\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\right\}\right)$.

Example 5.11 $F(t_1,...,t_6)=t_1-\varphi\left(\max\left\{t_2,\frac{t_3+t_4}{2},\frac{t_5+t_6}{2}\right\}\right)$.

Example 5.12 $F(t_1,...,t_6)=t_1-\varphi(\max\{t_2,\sqrt{t_3t_4},\sqrt{t_3t_5},\sqrt{t_4t_6},\sqrt{t_5t_6}\})$

Example 5.13 $F(t_1,...,t_6)=t_1-\varphi(at_2+bt_3+ct_4+dt_5+et_6)$, *where* $a,b,c,d,e\geq 0$ *and* $a+b+c+d+e<1$.

Example 5.14 $F(t_1,...,t_6)=t_1-\varphi\left(at_2+b\frac{\sqrt{t_5t_6}}{1+t_3+t_4}\right)$, *where* $a,b\geq 0$ *and* $a+b<1$.

Example 5.15

$F(t_1,...,t_6)=t_1-\varphi\left(at_2+b\max\{t_3,t_4\}+c\max\left\{\frac{t_3+t_4}{2},\frac{t_5+t_6}{2}\right\}\right)$, *where* $a,b,c\geq 0$ *and* $a+b+c<1$.

Example 5.16

$F(t_1,...,t_6)=t_1-\varphi\left(at_2+b\max\left\{\frac{2t_4+t_5}{3},\frac{2t_4+t_6}{3},\frac{t_3+t_5+t_6}{3}\right\}\right)$, *where* $a,b\geq 0$ *and* $a+b<1$.

By Theorem 4.2 and Example 5.9 we obtain

Theorem 5.9 *Let A,B,S and T be self mappings of a G - metric space (X,G) such that*

$$\psi(G(Ax,By,By))\leq\varphi(\max\{\psi(G(Sx,Ty,Ty)),\psi(G(Sx,Sx,Ax)),$$
$$\psi(G(Ty,By,By)),\psi(G(Sx,By,By)),\psi(G(Ax,Ty,Ty))\}),$$

for all $x, y \in X$, $\varphi \in \phi$ *and* ψ *is an almost altering distance.*

If (A, S) *and* (B, T) *satisfy* $CLR_{(S,T)}$ *- property, then*

i) $C(A, S) \neq \varnothing$,

ii) $C(B, T) \neq \varnothing$.

Moreover, if (A, S) *and* (B, T) *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point.*

By Theorem 5.9 and Theorem 5.3 we obtain

Theorem 5.10 *Let* A, B, S *and* T *be self mappings of a* G *- metric space* (X, G) *such that*

$$\int_0^{G(Ax,By,By)} h(t)dt \leq \varphi(\max\{\int_0^{G(Sx,Ty,Ty)} h(t)dt, \int_0^{G(Sx,Sx,Ax)} h(t)dt,$$
$$\int_0^{G(Ty,By,By)} h(t)dt, \int_0^{G(Sx,By,By)} h(t)dt, \int_0^{G(Ax,Ty,Ty)} h(t)dt)$$

for all $x, y \in X$, $\varphi \in \phi$ *and* $h(t)$ *as in Theorem 5.1.*

If (A, S) *and* (B, T) *satisfy* $CLR_{(S,T)}$ *- property, then*

i) $C(A, S) \neq \varnothing$,

ii) $C(B, T) \neq \varnothing$.

Moreover, if (A, S) *and* (B, T) *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point.*

Remark 5.3 *By Examples 5.10 – 5.16 we obtain similar results.*

If $\psi(t) = t$, by Theorem 5.9 we obtain

Theorem 5.11 *Let* A, B, S *and* T *be self mappings of a* G *- metric space* (X, G) *such that*

$$G(Ax, By, By) \leq \varphi(\max\{G(Sx, Ty, Ty), G(Sx, Sx, Ax),$$
$$G(Ty, By, By), G(Sx, By, By), G(Ax, Ty, Ty)\}),$$

for all $x, y \in X$ *and* $\varphi \in \phi$.

If (A, S) *and* (B, T) *satisfy* $CLR_{(S,T)}$ *- property, then*

i) $C(A, S) \neq \varnothing$,

ii) $C(B, T) \neq \varnothing$.

Moreover, if (A, S) *and* (B, T) *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point.*

Corollary 5.1 (**Theorem 2.2** (Giniswamy and Maheshwari, 2014)) *Let* (X, G) *be a symmetric* G *- metric space and* A, B, S *and* T *four self mappings of* X *such that*

1) (A, S) *and* (B, T) *satisfy* $CLR_{(S,T)}$ *- property,*

2) $G(Ax, By, Bz) \leq \varphi(\max\{G(Sx, Ty, Tz), G(Sx, By, Bz), G(Ty, By, Bz), G(By, Ty, Tz)\})$,

for all $x, y, z \in X$ *and* $\varphi \in \phi$,

3) (A,S) *and* (B,T) *are weakly compatible.*

Then A,B,S *and* T *have a unique common fixed point.*

Proof. If $y = z$, by 2) we have

$$G(Ax,By,By) \le \varphi(\max\{G(Sx,Ty,Ty),G(Sx,By,By),G(Ty,By,By),G(By,Ty,Ty)\}).$$

Since (X,G) is symmetric and φ is non decreasing, then

$$\begin{aligned} G(Ax,By,By) &\le \varphi(\max\{G(Sx,Ty,Ty),G(Sx,By,By),G(Ty,Ty,By)\}) \\ &\le \varphi(\max\{G(Sx,Ty,Ty),G(Sx,Sx,Ax),G(Ty,By,By), \\ &\quad G(Sx,By,By),G(Ax,Ty,Ty)\}), \end{aligned}$$

and by Theorem 5.11, A,B,S and T have a unique common fixed point.

5.4. Fixed Points for (φ,ψ) - Weakly Contractive Mappings in G - Metric Spaces

In 1997, Alber and Guerre-Delabriere (Alber and Guerre-Delabriere, 1997) defined the concept of weak contraction as a generalization of contraction and established the existence of fixed points for self mappings in Hilbert spaces. Rhoades (Rhoades, 2001) extended this concept in metric spaces. In (Beg and Abbas, 2006), the authors studied the existence of fixed points for a pair of (φ,ψ) - weakly compatible mappings.

New results are obtained in (Dorić, 2009; Raswan and Saleh, 2013) and in other papers.

The study of common fixed points of (φ,ψ) - weakly contractions with (EA) - property is initiated in (Sintunavarat and Kumam, 2011).

Also, some fixed point theorems for mappings with common limit range property satisfying (φ,ψ) - weakly contractive conditions are proved in (Imdad and Chauhan, 2013) and in other papers.

Definition 5.2

1) *Let* Ψ *be the set of all functions* $\psi:[0,\infty)\to 0,\infty)$ *satisfying*

a) ψ *is continuous,*

b) $\psi(0)=0$ *and* $\psi(t)>0$, $\forall t>0$.

2) *Let* Φ *be the set of all functions* $\phi:[0,\infty)\to 0,\infty)$ *satisfying*

a) ϕ *is lower semi - continuous,*

b) $\phi(0)=0$ *and* $\phi(t)>0$, $\forall t>0$.

The following functions $F:\mathbb{R}_+^6\to\mathbb{R}$ satisfy conditions $(F_1),(F_2)$ and (F_3).

Example 5.17 $F(t_1,...,t_6)=\psi(t_1)-\psi\left(\max\left\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\right\}\right)+\phi(\max\{t_3,t_4,t_5,t_6\})$.

Example 5.18

$$F(t_1,...,t_6) = \psi(t_1) - \psi(\max\{t_2,t_3,t_4,t_5,t_6\}) + \phi\left(\max\left\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\right\}\right).$$

Example 5.19

$$F(t_1,...,t_6) = \psi(t_1) - \psi\left(\max\left\{t_2,\frac{t_3+t_4}{2},\frac{t_5+t_6}{2}\right\}\right) + \phi(\max\{t_2,t_3,t_4,t_5,t_6\}).$$

Example 5.20

$$F(t_1,...,t_6) = \psi(t_1) - \psi\left(\max\left\{t_2,\frac{t_3+t_4}{2},\frac{t_5+t_6}{2}\right\}\right) + \phi\left(\max\left\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\right\}\right)$$

Example 5.21

$$F(t_1,...,t_6) = \psi(t_1) - \psi\left(\max\left\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\right\}\right) + \phi(\max\{\sqrt{t_3t_6},\sqrt{t_2t_5},\sqrt{t_5t_6}\})$$

Example 5.22

$$F(t_1,...,t_6) = \psi(t_1) - \psi(\max\{\sqrt{t_3t_6},\sqrt{t_2t_5},\sqrt{t_4t_6}\}) + \phi(\max\{t_2,t_3,t_4,t_5,t_6\})$$

Example 5.23

$$F(t_1,...,t_6) = \psi(t_1) - \psi\left(\frac{\sqrt{t_3t_6}+\sqrt{t_4t_5}+\sqrt{t_2t_6}}{1+\sqrt{t_3t_4}+\sqrt{t_4t_6}+\sqrt{t_2t_3}}\right) + \phi(\max\{t_2,t_3,t_4,t_5,t_6\})$$

By Theorem 4.3 and Example 5.17 we obtain

Theorem 5.12 *Let A, B, S and T be self mappings of a G - metric space (X, G) such that*

$$G(Ax, By, By) \le \psi(M_1(x, y)) - \phi(M_2(x, y)),$$

for all $x, y \in X$, where

$$M_1(x, y) = \max\{G(Sx, Ty, Ty), G(Sx, Sx, Ax), G(Ty, By, By), \frac{G(Sx, By, By) + G(Ax, Ty, Ty)}{2}\},$$

$$M_2(x, y) = \max\{G(Sx, Sx, Ax), G(Ty, Ty, By), G(Sx, By, By), G(Ax, Ty, Ty)\},$$

$\psi \in \Psi$ *and* $\varphi \in \phi$.

If (A, S) and (B, T) satisfy $CLR_{(S,T)}$ - property, then

i) $C(A, S) \neq \emptyset$,

ii) $C(B, T) \neq \emptyset$.

Moreover, if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point.

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TEOREME DE PUNCT FIX PENTRU DOUĂ PERECHI DE FUNCŢII CU PROPRIETATEA LIMITEI COMUNE ÎN SPAŢII *G* – METRICE

(Rezumat)

Scopul acestei lucrări este demonstrarea unei teoreme de punct fix pentru două perechi de funcţii în spaţii *G* - metrice, care să generalizeze rezultatele din (Popa şi Patriciu, 2014) şi să unifice rezultatele din (Giniswamy şi Maheshwari, 2014). De asemenea, este obţinut un rezultat nou pentru un şir de funcţii. În ultima parte a lucrării, ca aplicaţii, sunt obţinute câteva rezultate de punct fix pentru funcţii care satisfac o condiţie contractivă de tip integral, pentru funcţii aproape contractive, pentru funcţii ϕ – contractive şi (ϕ,ψ) – contractive în spaţii *G* – metrice.

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TOTAL DOSE RELATED TO TUMOR VOLUME AND TOXICITY RISK CORRELATION IN MODERN RADIOTHERAPY

BY

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Abstract. Radiotherapy is a critical and inseparable component of comprehensive cancer treatment and care. It is estimated that about 70% of cancer patients would benefit from radiotherapy for treatment of localized disease, local control, and palliation. Yet, in planning and building treatment capacity for cancer, radiotherapy is frequently the last resource to be considered.

Keywords: radiotherapy; tumor volume; dose; radiation toxicity.

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1. Introduction

Managing cancer requires both effective preventive measures to reduce future burden of disease, and health-care systems that provide accurate diagnose and high-quality multimodality treatment. Such multimodality treatment should include radiotherapy, surgery, drugs, and access to palliative and supportive care. Radiotherapy is perceived as a complex treatment. Estimation of the exact proportion of new cancer cases that will need radiotherapy is complex, in view of the variable patterns of cancer presentation and limited information on the current proportion of patients receiving radiotherapy. During the past 20 years, several investigators have developed evidence-based estimates of desirable radiotherapy use on the basis of the indication for radiotherapy in clinical practice guidelines and the distribution of cancer and different stages of disease at presentation. These estimates suggest that 60-70% of all patients with cancer will need radiotherapy. Radiation therapy acts both on tumor cells and normal tissue making the therapeutic benefit both toxicities and complications caused by acute and delayed treatment. Maintaining the balance between local tumor control and minimize side effects and complications remains a challenge for radiotherapy. Unfortunately, despite significant technological advances of the past three decades, more than 100 years of experience in radiotherapy, indicates that data on the effects of radiation are beneficial and detrimental in many cases.

In historical perspective the first comments on the biological effects of radiation from the late IXX century belong to Gassmann (1898), which depicts two histological types of ray-induced chronic ulcer. The first study analyzing tolerances healthy tissues to radiation therapy has been published by Rubbin and Casarett treaty “Radiation Clinical Pathology” (1968). The paper presents a set of pictures taken during irradiation, highlighting the progression of lesions radiomucositis and described the evolution from acute to chronic and tardive. 80-90 years of the twentieth century have made significant progress by introducing radiotherapy CT simulators, computer systems dosimetry of collimator and multi optimizations that allowed the transition to three-dimensional radiotherapy volumes enabling evaluation of receiving certain doses. There were also introduced unique criteria for assessing the level of toxic effects of radiation in the form of scales, the LENT-SOMA being used and CTCAE (Dobbs *et al*., 2009).

The first database, with correlations between organ volume receiving a given dose risk of complications is offered by prestigious study Emami (1991). It proposes dividing the organ on the basis of volumetric three recommendations restrictions being given doses 1/3, 2/3 and full organ. Original work, known as Emami Guide, was, despite its limitations, a review of medical literature until 1991. It is only for severe complications. 3D techniques, IMRT, VMAT were nonexistent at the time, so was used only conventional

fractionation (2Gy/fraction). In the 25 years since the publication of his work Emami, the practice has been completely revolutionized radiotherapy (Bortfeld *et al*., 2006; Van der Kogel and Joiner, 2009):

- multi-disciplinary cancer treatment become standard;
- end-points in the complications have changed;
- 3D-CRT and "inverse planning" totally replaced the 2D radiation therapy;
- CT simulation images using CT, MRI and PET-CT become standard

As a result, the dose distribution has become increasingly more complex and more recently, was placed 4-dimension (time). It became necessary to introduce new updated models correlation dose-volume-complications. The Quantec work, resulting a collective effort by 57 experts, appears to support ASTRO (American Association of Radiotherapy) and AAPM (American Association of Medical Physics), and is published in the Supplement to the journal "*International Journal of Radiation Oncology, Biology, Physics*" (the Red Journal), Vol. 76, No. 3, 2010 (Nishimura and Komaki, 2015). This gives the review last 2 decades radiotherapy putting in relationships, in a detailed way, the parameters dose/volume with clinical complications. It also provides a simple set of data grouped into 16 radiosensitive organs in order to provide a useful and easily accessible to validate plans carried out jointly by the radiotherapists, physicians and medical physicists (Van der Kogel and Joiner, 2009; Nishimura and Komaki, 2015).

In an era of personalized medicine, progress means that radiotherapy beams can be shaped and modulated to conform to the exact shape of tumors, maximizing radiation dose deposition in the cancer while sparing normal tissues from high doses, those most likely to evoke normal tissue toxic effects. Radiotherapy is also a powerful instrument in palliation of symptoms associated with cancer. According to the survey noted, factors affecting normal tissues to radiation tolerance are:

- patient condition (age, comorbidities, Karnofsky score, pathogens, response to therapy);
- organ radio sensibility variations;
- serial dose-response organization (spinal cord);
- organization of parallel volume effect (liver, lung);
- serial and parallel mixed organization (kidney);
- natural history of the tumor;
- radio therapeutic treatment: dose value (maximum, medium, minimum dose), dose, overall treatment time, energy, irradiated volume;
- non-radio therapeutic treatment: chemotherapy, surgery, *i.e*.

In the context of the plurality of data from the medical literature, it aims to develop predictive models based on the dose-volume, which will act as a guide only and may not substitute medical experience.

With the development of mathematical models and radiobiological, more and more authors use conversion dose/fraction, at a dose equivalent biological dosimetry to compare different parameters. Izo-effect formula (1) based on the linear quadratic model and the index α/β is calculated from survival curves cell tumor model extrapolated to five.

$$BED = \frac{E}{\alpha} = D\left(1 + \frac{d}{(\alpha+\beta)}\right) \quad (1)$$

Failure assessment values α/β in human tumor tissue makes use of radiobiological model, with more than indicative value, cannot be recommended as routine practice. Applying value BED (2) or 2Gy equalization formula should be implemented taking into account the limits of the model

$$EQD_2 = D\left(1 + \frac{d+(\propto/\beta)}{2Gy+(\propto/\beta)}\right) \quad (2)$$

and certain physical and biological parameters that were taken into account in the work underlying the guidelines dosimetric (Van der Kogel and Joiner, 2009):

- dose/fraction has a significant impact in the acute and late complications;
- 1.8 or 2Gy/fraction /5 fractions/ week is considered standard fractionation;
- most publications of the last two decades considered the report of α/β = 2 for CNS;
- BED Quantec publications calculated using a value of α/β = 3 for CNS;
- IMRT technology allows the use of any fractional (integrated boost) that makes it difficult to evaluate existing plans after recommendations.

With broad deployment IMRT and VMAT techniques, Niemierko proposed a biological model for assessing treatment plans that would be applicable to non-uniform dose distributions. At its core are the parameters EUD (equivalent uniform dose transmitted tissue would produce the same effect on cell populations) and NTCP (healthy tissue likelihood of developing complication) (Schwartz *et al*., 2005; Rubin *et al*., 2014). NTCP use in clinical practice is recommended only as a guide, new studies are needed to validate this parameter as a predictor of toxicities.

A. Central Nervous System (CNS) & Sensorial Organs

1. Brain tissue. Brain tissue radiation toxicity is the neurocognitive impairment and cerebral radionecrosis. This generally occurs between three months and several years (average 1-2) from irradiation (Hayes and Kruger, 2007).

Volume	Dose	Risk of Radionecrosis
1/3cerebral volume	D < 60Gy	5% (Emami *et al*., 1991)
	D max < 60Gy	3%
	D max = 70Gy	5%
	D max = 90Gy	10%
$\alpha/\beta = 3$ BED	D = 120Gy	5%
SRT	D > 12Gy	20%
children	D total (WBRT) > 18Gy	Neurocognitive modifications
Re-irradiation $\alpha/\beta = 2$ (2Gy equivalent)	D total < 100Gy	

Risk Factors (Bentzen *et al*., 2010; Marks, 2010a; Marks, 2010b):

- old age / young (children);
- female gender;
- NF-1 mutation;
- extensive surgery;
- diabetes;
- hydrocephalus;
- chemotherapy (especially with methotrexate);
- dose/fractionation/volume;
- a low index of conformity;
- location of the target volume.

2. Brainstem. Induced toxicity on the brainstem can be debilitating and potentially lethal due to its origin at this level of the 12 pairs of cranial nerves:

Volume	Dose	Toxicity risk (%)
100% brainstem	< 50Gy	5% (Emami *et al*., 1991)
100% brainstem	< 54Gy	5%
V < 1-10 cm^3	< 59Gy	< 5%
V < 1 cm^3	< 64Gy	< 5%
SRT	D max > 12.5Gy	

Risk Factors (Bentzen *et al*., 2010; Lawrence *et al*., 2010):

- hypertension;
- diabetes;
- number surgery;
- target volume in proximity;
- MRI imaging for a lack of planning.

3. Spinal cord. Bone marrow toxicity of radiation is rare but severe consequences (paralysis, sensory deficit, pain, urinary incontinence). Toxicities were evaluated doses of 2-9Gy /fraction, calculating the equivalent dose of 26Gy to a value $\alpha/\beta = 0.87$ (Dawson *et al*., 2010; Emami, 2013).

Risk factors (Bentzen *et al*., 2010; Mayo *et al*., 2010a; Mayo *et al*., 2010b):

- neurotoxic chemotherapy;
- segment irradiated bone marrow (cervical bone is more sensitive than chest probably the components of cranial nerves - IX, X, XI, XII);
- young age (children).

Volume	Dose	Risk for myelopathy (%)
	D max = 50Gy	0.2%
	D max = 60Gy	6%
	D max = 69.6%	50%
SRT – unique dose	D max = 13Gy	1%
SRT – hyper fractions	D max = 20.6Gy	1%
Re-irradiation	25% dose “forgotten” after 6 months	

4. Optic nerves & optic chiasma. Optic neuropathy is rare and is manifested by rapid and painless loss of vision (Van der Kogel and Joiner, 2009; Kirkpatrick *et al*., 2010).

Volume	Dose	Risk for Optic neuropathy (%)
Whole volume organ	D < 50Gy	
	D max = 54Gy	< 3%
	D max = 55-60Gy	> 3-7%
	D max = 60Gy	> 7-20%

Risk Factors (Bentzen *et al*., 2010; Kirkpatrick *et al*., 2010):

- age;
- diabetes;
- hypertension;
- chemotherapy(anticancer agent - Bevacizumab has a protective effect);
- re-irradiation (dose fraction within the first irradiation).

5. Retina. Radiation induced retinopathy is a decrease in visual acuity similarly to diabetic retinopathy. There were reported rarely retinopathy radiation induced at doses below 50Gy, but for doses < 45Gy received by posterior pole, it is practically non-existent (Dobbs *et al*., 2009; Van der Kogeland Joiner, 2009).

Risk Factors (Bentzen *et al*., 2010; Bhandare *et al*., 2010):

- hypertension;
- diabetes;
- dose/volume/fractionation (to 3-fold decrease in the risk of retinopathy by hyper fraction).

6. Cochlea. Damage of cochlea consists in neurosensorial hearing loss. High frequency hearing impairment is more common than at low frequencies. Age and high acuity hearing before treatment and chemotherapy with Cisplatin are factors that significantly affect toxicity. Occurrence of otitis media after radiotherapy is considered a significant factor (Bentzen *et al*., 2010; Deasy *et al*., 2010).

Volume	Dose	Neurosensorial risk (%)
concomitant with cisplatin	D < 45Gy	< 30%
	D med < 47Gy	< 15%
SRT	D max < 14Gy	< 25%

Risk factors (Dobbs *et al*., 2009; Bentzen *et al*., 2010; Deasy *et al*., 2010):

- total dose of irradiation;
- age;
- positioning a target volume;
- dose of cisplatin
- hearing aid existing pathologies and subsequent irradiation.

B. Head & Neck

1. Parotids, submandibular and sublingual salivary glands. Impaired secretion of salivary glands (xerostomia) is common for cephalic extremity irradiation and can be a cause of deteriorating quality of life patient for a period of up to 2 years after completion of radiotherapy. Xerostomia is to reduce salivary flow and significantly reduces its risk by reducing the dose from a single submandibular gland (recommended doses < 35Gy). Xerostomia grade IV (decrease by more than 75% of salivary volume) was the threshold for who proposed building dosimetry and is a risk factor for oral bacterial and fungal superinfections after radiotherapy (Dobbs *et al*., 2009; Rancati *et al*., 2010).

Volume	Dose	Risk for Xerostomia (%)
Bilateral parotids	D med < 25Gy	< 20%
Unilateral parotid	D med < 20Gy	< 20%

Risk Factors (Bentzen *et al*., 2010; Marks *et al*., 2010a; Marks *et al*., 2010b):

- drugs that interferes with salivation;
- eating disorders;
- rheumatologic diseases;
- smoking.

2. Mandible. Rates of osteonecrosis of the jaw has dropped considerably with the introduction of IMRT and VMAT techniques (Dobbs *et al*., 2010; Marks *et al*., 2010a; Marks *et al*., 2010b).

Dose	Risk for Osteonecrosis (%)
D max < 70Gy	< 5%

Risk Factors (Bentzen *et al*., 2010):

- radiation dose;
- chemotherapy;
- dental hygiene;
- tumor site;
- oro-maxillo-facial surgery history.

3. Pharyngeal constrictors muscles. Dose escalation irradiation for head and neck cancers has increased the rate of late toxicities (dysphagia and aspiration) on swallowing mechanisms. Some studies have associated toxicity with the dose received by superior and medium pharyngeal constrictor muscles, others studies considered relevant only the dose received by inferior pharyngeal constrictor muscle (Kavanagh *et al*., 2010).

Dose	Toxicity risk (%)
D_{medie} < 50Gy	20%
D_{max} < 70Gy	< 5% (compulsory PEG, aspiration)

Risk Factors (Bentzen *et al*., 2010; Marks *et al*., 2010a; Marks *et al*., 2010b):

- local advanced neoplasms;
- concomitant chemotherapy (hazard of swelling and dysphagia).

4. Larynx. Radiation toxicity affecting the larynx include laryngeal edema formation and (especially glottis). Radionecrosis laryngeal cartilages risk is low in the context of using modern techniques, but remains present in particular as a consequence the long term (Marks *et al*., 2010a; Marks *et al*., 2010b).

	Dose	Toxicity risk (%)
RTE +CHT	D_{max} < 66Gy	< 20% (dyspnea)
RTE +CHT	D_{max} < 50Gy	< 30% (aspiration risk)
	D_{medie} < 44Gy	< 20% (edema)

Risk factors (Dobbs *et al*., 2009; Bentzen *et al*., 2010; Marks *et al*., 2010a; Marks *et al*., 2010b; Michalski *et al*., 2010; Pan *et al*., 2010):

- concurrent chemotherapy;
- staging (except T1, larynx glottis → low risk of impaired phonation);
- concomitance with EGFR inhibitors (cetuximab) → mucositis/infections.

C. Thorax

1. Brahial plexus. Brachial plexopathy may be manifested by pain, paresthesia or upper limb motor deficit. Muscular atrophy and edema are occasional complications. Toxicity can signal and after 5 years of the end of radiotherapy (Dobbs *et al*., 2009; Van der Kogel and Joiner, 2009; Roach *et al*., 2010; Viswanathan *et al*., 2010).

Volume	Dose	Risk of plexopathy (%)
Whole brachial plexus	D max < 60Gy	< 5%

Risk Factors (Bentzen *et al*., 2010):

- hyper fractionated regimes;
- Lymphadenectomy;
- obesity;
- hypertension;
- diabetes
- valvulopathy.

2. Lungs. Radice pneumonitis is one of the most common toxicities in patients receiving radiation for lung neoplasms: breast, esophagus and mediastinal lymphadenopathy. The risk of developing pneumonitis radice limited dosage used in treating these malignancies (Van der Kogel and Joiner, 2009; Gagliardi *et al*., 2010; Werner-Wasik *et al*., 2010).

Volume	Dose	Pneumonitis radice risk (%)
V_5 < 42%	D med = 7Gy	5%
V_{20} < 22%	D med < 13Gy	10%
V_{20} < 31%	D med < 20Gy	20%
V_{20} < 40%	D med < 24Gy	30%
	D med < 26Gy	40%

Risk factors (Bentzen *et al*., 2010; Gagliardi *et al*., 2010):

- chemotherapy with taxanes, gemcitabine;
- concomitant therapy with TKI inhibitor (erlotinib);
- pre-existing lung diseases

3. Heart and pericardium. Pericarditis and cardiac mortality in the long run are two of the most common toxicities. Increase in survival for patients with breast cancer and lymphoma requires revaluation heart of the doses received and their correlation with late mortality.

	Volume	Dose	Toxicity risk (%)
RTE +Adriamicina	3/3 heart $V_{25} < 10\%$	D < 15Gy	1% risk 15 years after the end of irradiation
RTE +Adriamicina	3/3 heart $V_{30} < 46\%$	D < 30Gy	Risk < 15% (pericarditis)

Risk factors:

- age;
- sex;
- diabetes;
- hypertension;
- high levels of cholesterol;
- smoking;
- family history of heart.

4. Esophagus. Radice esophagitis is constant during irradiation of thoracic tumors, and is manifested by dysphagia, swallowing and may adversely affect the patient's condition causing discontinuation of treatment.

Volume	Dose	Risk of radice esophagitis (%)
$V_{35} < 50\%$	D med < 34Gy	Grd III = 5-20%
$V_{50} < 40\%$		Grd II < 20%
$V_{70} < 20\ \%$		

Risk factors:

- aged > 70 years;
- hyper fractionated regimes;
- concomitant boost;
- concurrent chemo-radiotherapy;
- large number of hotspots in the treatment plan.

D. Abdomen

1. Liver. Radio-induced hepatitis usually occurs between 2 weeks and 3 months after completion of radiation therapy, the radiation dose limiting complication of biliary tumors and upper digestive tract. Subacute form of hepatitis is usually manifested by fatigue, abdominal pain, hepatomegaly, ascites anicteric, increased alkaline phosphatase and liver enzymes.

Volume	Dose	Hepatitis risk (%)
Liver cancer with preexisting disease	D med < 30Gy D med < 28Gy	5%
Whole organ	≤ 30Gy (2Gy/fr) ≤ 21Gy (3Gy/fr) < 28Gy (2Gy/fr) < 21Gy (3Gy/fr)	5%
	D med < 42Gy	
Liver metastasis	D med < 13Gy (3fr) D med < 18Gy (6fr)	< 5%

Risk factors:

- hepatocarcinoma > metastases;
- hepatitis B and C;
- portal thrombosis;
- chemotherapy;
- chemoembolization;
- tumor stage;
- male gender;
- score Child - Pugh.

2. Stomach. Late toxicity manifests as gastric ulceration and dyspepsia. Loss of appetite, feeding behavior and disturbances in fluid intake can lead to malnutrition and cachexia, exacerbating the patient's condition.

	Volume	Dose	Risk of gastric toxicity (%)
	3/3 stomach	D < 50Gy	
SRT	V 22.5 < 4% / 5 cm^3	D max < 30Gy (3Gy/fr)	5-7%

3. Small intestine. Gastro-intestinal toxicity is significantly increased in case of concurrent chemotherapy or previous abdominal surgery. Decrease of absorption, diarrhea, impaired intestinal flora and pathogens are frequent

complications during irradiation for abdominal and pelvic tumors. New studies show that large volumes of small intestine receiving relatively low doses are correlated with acute toxicity. If the individual emerges intestines, the most representative volume predictor of toxicity is V15. Late toxicity consists of obstructions, perforations and is commonly associated with abdominal wall surgery.

Organ	Volume	Dose	Risk of enteric toxicity (%)
Intestinal coils	$V_{15} < 150$ cm^3	D < 50Gy	10%
Peritoneal cavity	$V_{45} < 195$ cm^3	D < 50Gy	10%
1/3 small intestine	$V_{50} < 51\%$	D < 50Gy	
SRT	$V_{12.5} < 30$ cm^3	D max < 30Gy (3-5Gy/fr)	10%

Risk factors:

- anatomical conformation (large intestines in the field of radiation);
- abdominal surgery;
- cardiovascular pathologies;
- diabetes;
- chemotherapy (adriamycin, 5-FU);

E. Pelvis

1. Rectum. Improving regimens irradiation in prostate cancer with the decrease of late post-radiotherapy rectal toxicity has made many of these patients as long term survivors. Dose escalation, by moving from 2D and 3D techniques to IMRT required the assessment of dosimetric parameters correlated with late proctopathia.

Volume	Toxicity risk grd II (%)	Toxicity risk grd III (%)
$V_{50} < 50\%$	15%	10%
$V_{60} < 35\%$	15%	10%
$V_{70} < 20\%$	15%	10%
$V_{75} < 15\%$	15%	10%

Risk factors:

- diabetes;
- inflammatory digestive diseases;
- hemorrhoids;
- age;
- treatment with anti-androgens;
- size rectum;
- abdominal surgery.

2. Bladder elasticity makes difficult a performing dosimetric analysis with predictive toxicity. Affecting the entire body may be manifested by dysuria, urinary frequency, bladder spasm, reducing the flow urinary incontinence. Damage is focal manifestations: hematuria, fistula, obstruction, ulceration and necrosis.

Risk factors:

- hormone therapy;
- chemotherapy (cyclophosphamide);
- TUR-V&TUR-P;
- underlying genitourinary pathology;
- hysterectomy;
- obesity;
- smoking;
- black race;
- age;
- diabetes.

3. Kidney. Renal dysfunction after radiotherapy can cause symptoms and biochemical and radiological changes form. High latency ranges are as renal toxicity late to be undervalued. Most studies have evaluated serum creatinine clearance decreased in relation to the dose received by both kidneys.

Risk factors:

- renal failure;
- diabetes;
- cardiac pathologies;
- smoking.

4. Penile bulb. Erectile dysfunction can be a cause of discomfort for patients with prostate cancer. The dose received by the penile bulb is considered a predictor.

Volume	Dose	Toxicity risk (%)
V_{60}-V_{70} < 70Gy	D med < 52Gy	< 55%
V_{90} < 50Gy	D med 95% din gland < 50Gy	< 35%

Risk factors:

- age;
- diabetes;
- treatment with anti-androgens;
- hypertension;
- smoking

F. Other Radiosensitive Organs

Radio-sensitive organs outside Quantec included in the guide, benefit the records of the toxic and other parts of the body. Keeping average dose associated with various complications, below the various studies, may help optimize quality of life. In clinical practice, to assess the dose equivalent hypo-fractionated regimes use the value ratio α/β = 10Gy to the tumor tissue and α/β = 3Gy for late toxic effects. For a more precise risk assessment of the possibility of toxic and tumor control is recommended in the report izo-equivalent formula α/β correlated with each organ specific toxicities

Legend

D max – maximum dose received by an organ;

D medium – average dose received by an organ;

Vx – The volume of the organ receiving the higher dose of "x" Gray;

Dy – minimum dose received by the 'y'% of an organ;

SRT – Stereotactic Radiotherapy;

WBRT – "Whole brain" Palliative Radiotherapy;

PEG – percutaneous gastrostomy;

IMRT – intensity modulated radiotherapy external;

VMAT – intensity modulated radiotherapy external volume (with continuous irradiation Rotational);

Anti - EGFR – epidermal growth factor inhibitor;

TKI – tyrozin kinase inhibitor;

5FU – 5-Fluorouracil;

ACE – inhibitor of angiotensin converting enzyme;

CT – computed tomography;

MRI (MRI) – magnetic resonance imaging;

REVERSE PLANNING – planimetric technique is proposed the conformation bundles computer after dosimetry constriction introduced by physicist;

Quantec – Quantitative Analyses of Normal Tissue Effects in the Clinic;

PET-CT – Positron emission tomography;

E – biological effect;

α/β – The ratio of intrinsic cellular radiosensitivity and cell fraction which completely repaired lesions in 6 hours or more;

EUD – equivalent uniform dose transmitted tissue would produce the same effect on cell populations;

NTCP – Probability healthy tissue of developing complications;

EQD2 – 2Gy fractionated dose equivalent that would produce the same biological effect as prescribed.

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DOZA TOTALĂ CORELATĂ CU VOLUMUL TUMORAL ŞI RISCUL DE TOXICITATE ÎN RADIOTERAPIA MODERNĂ

(Rezumat)

Radioterapia este o componentă esențială și inseparabilă în contextul tratamentului multidisciplinar al cancerului. Se estimează că aproximativ 70% dintre pacienții cu cancer ar putea beneficia de radioterapie pentru tratamentul bolii localizate, controlul local și paliativ. Cu toate acestea, în planificarea și implementarea secvențelor terapeutice oncologice, radioterapia este frecvent ultima resursă care se ia în considerare.

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DOSIMETRIC COMPARATIVE EVALUATION PARAMETERS FOR DIFFERENT RADIOTHERAPY TECHNIQUES (3D-CRT, IMRT, VMAT) IN PARANASAL SINUSES CANCERS TREATMENT

BY

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Abstract. Rinosinusal cancers accounts for about 3% of all malignancies, most developing to the maxillary sinuses (70%), followed by ethmoid (20%), frontal (3%) and sphenoidal (1%) sinuses. Anatomical position and late symptomatology in advanced stages make these malignancies difficult to diagnose, surgical approach and adjuvant treatment with radiation having a role in getting local control. Radiosensitive organs in proximity made difficult to deliver tumoricidal dose irradiation by conventional radiotherapy. Implementation of 3D-CRT technologies (3D conformal) based on the use of MLC (multi-leaf collimator) and then inverse planning techniques IMRT (intensity modulated radiation therapy) and VMAT (volumetric modulated arc

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therapy) resulted in dose reductions in OAR (organs at risk) and better dose homogeneity in PTV (planning target volume).

Keywords: radiotherapy; paranasal sinuses; IMRT; VMAT.

1. Introduction

Nasal cavity and paranasal sinuses cancers include tumors originate from the paranasal cavities (ethmoid, maxillary, frontal and sphenoidal) or from the nose (excluding nasal vestibule) and is a rare type of cancer, about 0.2-0.8% of all cancers and 5% of head and neck cancers. It is often late diagnosed with nonspecific clinical symptoms having high tumor aggressivity and a poor prognosis. Most commonly occurs in the maxillary sinus (70%), followed by ethmoid (20%), frontal (3%) and sphenoidal (1%) sinus (Jégoux *et al*., 2013). Surgical resection with negative margins followed by adjuvant radiotherapy is the optimal treatment. In some advanced cases surgical anatomical limits make impossible a complete resection, definitive radiotherapy with or without concurrent chemotherapy being the only therapeutic option. The challenge to deliver a tumoricidal dose on a relatively large volume in the immediate vicinity of radiosensitive critical organs (optic nerves, lenses, optic chiasma, brain) made necessary the development of new high precision methods in radiotherapy. Inverse planning techniques provides superior dose conformity compared to 2D and 3D radiotherapy often associated with high toxicity: radical cataract, dry eye syndrome caused by lacrimal gland function loss, retinopathy or even blindness caused by irradiation of optical aperture (optic nerve and chiasm). Non-coplanar IMRT technique can provide superiority in terms of organs at risk protection, especially for tumors of the nasal cavity and for target volume situated between the eyes. Implementation of rotational intensity modulated technique VMAT brings advantages over IMRT technique in particular by decreasing treatment time and number of monitor units (Bortfeld *et al*., 2006). The paper aims to benchmark target volume coverage and mean doses and Dmax (maximum doses) receive by organs at risk in case of neoplasm of maxillary sinus locally advanced, comparing alternative treatment plans IMRT and VMAT (two half arcs, single arc, double arc) (Jeong *et al.*, 2014).

2. Materials and Methods

We present a case of a locally advanced right maxillary sinus cancer who received definitive radiotherapy in total dose DT = 66Gy/33fr/PTV-T (3D-CRT technique). For a patient with an advanced right maxillary sinus cancer previously treated with 3D-CRT radiotherapy, IMRT and VMAT alternative plans were proposed (two half arcs, single arc and double arc)

comparing the dose to OARs, MU (number of monitor units) and target volume coverage. All plans offered doses in accepted limits for organs at risk with similar target volume coverage. VMAT technique offers the advantage of a short treatment time and is a feasible option for busy radiotherapy centers (Biagioli *et al*., 2007).

Patient immobilization was made using a thermoplastic mask and for target delineation volumes (GTV, CTV, PTV) was performed CT simulation, a rigid registration being made between the diagnosis and the simulation CT. Delineation of interest volumes, organs at risk and dosimetry calculation were performed by Eclipse Treatment Planning System™(TPS) software. Dosimetric evaluation of treatment plans took into account target coverage by the 95% isodose and doses received by organs at risk according to recommendations of Quantec and Emami papers (Miura *et al*., 2012). In order to verify the accuracy of the positioning, X-rays kV was performed weekly (every 5 fractions) from the treatment machine, a linear accelerator Varian Clinac iX with 120 multi-leaf collimator.

Subsequently alternative plans were proposed by coplanar IMRT and three different plans using VMAT different from each other by the angle described by the gantry (two half arcs, single or double arc) for a comparative dosimetric evaluation, reproducibility with plans being validated by ArcCHECK® platform (Figs. 1-3).

Conformity index-CI (ratio of volume surrounded by 95% isodose and the volume of PTV), homogeneity index-HI (the ratio of difference between volume which receives 2% and 98% of the prescribed dose and the volume surrounded by 50% isodose) for target volume (PTV), were evaluated together with mean dose (Dmean), maximum dose (Dmax) and the number of monitor units for each technique received by the OARs.

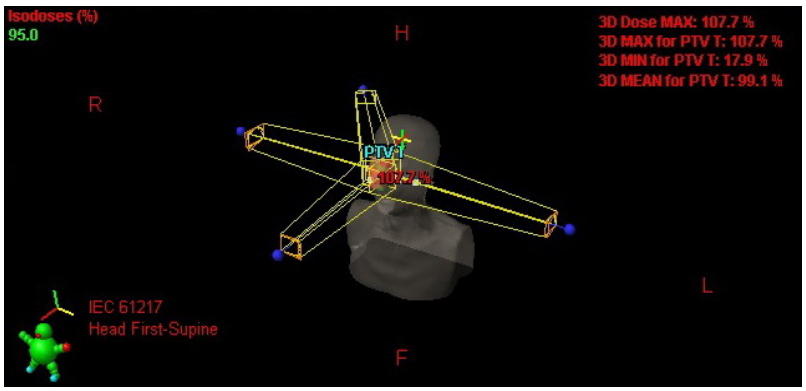

Fig. 1 – Beam orientation for 3D-CRT plan.

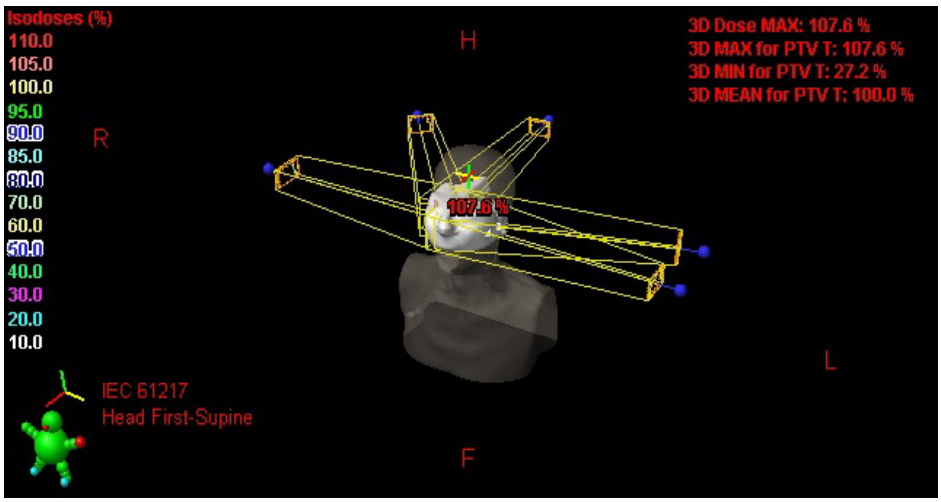

Fig. 2 – Beam orientation for IMRT plan.

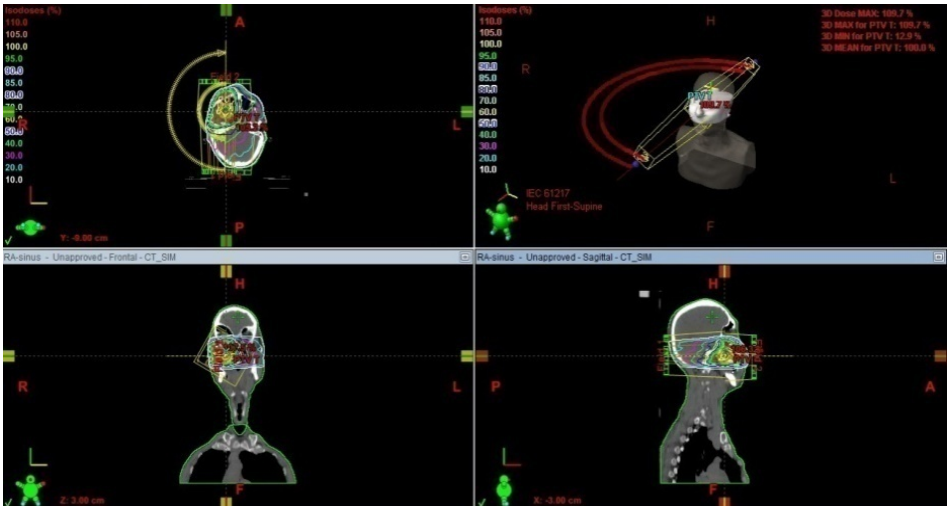

Fig. 3 – Half arc radiotherapy plan – beam angle rotation and isodose curves.

3. Results

All doses received by organs at risk using IMRT and VMAT techniques (two half arcs, single arc, double arc) were compared to the dose received by the same organs in 3D-CRT technique (Fig. 4).

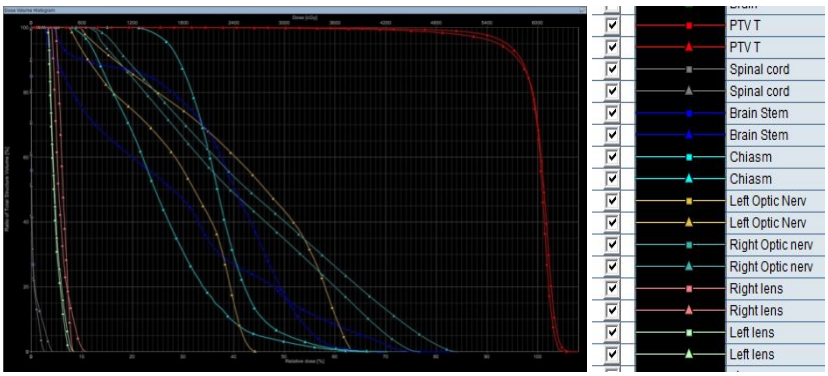

Fig. 4 – DVH comparison for OAR and PTV between IMRT and VMAT- 2 arcs plans.

IMRT method significantly reduces the mean dose received by spinal cord (46.77%) but significantly increase the dose to the right optic nerve (70.03%). The VMAT methods (two half arcs, single arc, double arc) shows the disadvantage of significantly increasing the dose received by the brainstem with 48.28%, 55.61%, 60.59%, optic nerve as with 65.16%, 79.93%, 84.93% and optic chiasm 62.36%, 57.50% and 31.64% (Table 1).

Table 1
Absolute (cGy) and Relative Dmean for OAR Reported to 3D-CRT

	Absolute number of MU					Relative number of MU			
Radiotherapy Technique	3D - CRT	IMRT	VMAT- 1/2 arc	VMAT- 1 arc	VMAT- 2 arcs	IMRT (%)	VMAT- 1/2 arc (%)	VMAT- 1 arc (%)	VMAT- 2 arcs (%)
Spinal cord	38.70	20.60	32.30	35.30	31.90	-46.77	-16.54	-8.79	-17.57
Brainstem	1403.20	1715.10	2080.70	2183.50	2253.40	22.23	48.28	55.61	60.59
Brain	502.90	547.40	357.70	738.00	739.40	8.85	-28.87	46.75	47.03
Left eye	769.40	789.70	517.00	553.80	605.30	2.64	-32.80	-28.02	-21.33
Left lens	322.40	284.20	289.60	290.90	294.90	-11.85	-10.17	-9.77	-8.53
Right eye	1488.30	1321.80	1047.30	1022.70	972.90	-11.19	-29.63	-31.28	-34.63
Right lens	329.90	354.20	325.30	368.50	382.00	7.37	-1.39	11.70	15.79
Right optic nerve	1443.60	2454.60	2384.30	2597.50	2669.70	70.03	65.16	79.93	84.93
Left optic nerve	1920.40	2541.30	1496.60	1656.40	1740.60	32.33	-22.07	-13.75	-9.36
Optic chiasma	1702.30	1546.00	2763.90	2681.10	2240.90	-9.18	62.36	57.50	31.64

All VMAT methods decrease the mean dose to the spinal cord, contralateral eye and contralateral lens. The maximum dose is reduced or almost equal for all OARs except spinal cord in which significant increases were observed (57.87%, 62.53%, 37.57%).

IMRT technique significantly increases the number of M.U. compared to the number of M.U. delivered by 3D-CRT (128%). VMAT techniques (two half arcs, single arc, double arc) decrease the number of MU with 31.67%, 27.67%, 30.67% (Table 2).

Table 2
Absolute and Relative Number of MU Reported to 3D-CRT

	Absolute number of MU					Relative number of MU			
Radiotherapy technique	3D	IMRT	VMAT-1/2 arc	VMAT-1 arc	VMAT-2 arcs	IMRT (%)	VMAT-1/2 arc (%)	VMAT-1 arc (%)	VMAT-2 arcs (%)
MU	300.00	684.00	395.00	383.00	392.00	128.00	31.67	31.67	30.67

CI closest to the optimum value "1" is obtained with IMRT techniques and VMAT and HI is closest to the optimum value "0" technique IMRT (Table 3).

Table 3
Absolute (cGy) and Relative Dmax for OAR Reported to 3D-CRT

	Absolute number of MU					Relative number of MU			
Radiotherapy Technique	3D - CRT	IMRT	VMAT-1/2 arc	VMAT-1 arc	VMAT-2 arcs	IMRT (%)	VMAT-1/2 arc (%)	VMAT-1 arc (%)	VMAT-2 arcs (%)
Spinal cord	195.10	169.00	308.00	317.10	268.40	-13.38	57.87	62.53	37.57
Brainstem	4791.5	4965.8	4091.4	4089.7	4191.3	3.64	-14.61	-14.65	-12.53
Brain	6462.2	6132.1	6581.0	6115.9	6075.4	-5.11	1.84	-5.36	-5.99
Left eye	3365.6	3645.4	1564.1	1882.7	2143.7	8.31	-53.53	-44.06	-36.31
Left lens	574.10	469.90	495.70	504.60	515.90	-18.15	-13.66	-12.11	-10.14
Right eye	6105.5	6026.7	5870.7	5648.1	5605.0	-1.29	-3.85	-7.49	-8.20
Right lens	608.10	657.40	463.40	505.00	505.60	8.11	-23.80	-16.95	-16.86
Right optic nerve	3392.7	4609.6	4862.6	5130.8	5065.0	35.87	43.33	51.23	49.29
Left optic nerve	3303.8	3803.1	2393.5	2508.3	2681.1	15.11	-27.55	-24.08	-18.85
Optic chiasma	4761.3	4026.3	4569.6	4482.5	4226.4	-15.44	-4.03	-5.86	-11.23

4. Discussion

In head and neck cancer radiotherapy dosimetry, a 43-45Gy constriction spinal cord in order to reduce the risk of radio-induced myelopathy, limits delivered dose in the target volume during conventional radiation therapy. Inverse planning techniques made possible the simultaneous irradiation with different fractionations and different doses for different volumes allowing dose escalation in the areas of tumor radio-resistance. The inclusion of functional

imaging PET-CT and diffusion MRI combined with a high-resolution structural imaging could bring benefit in dose escalation.

Basic treatment leads to local failure in 70%. Salvage therapy has a success rate of 30-40% off in head and neck cancers but few patients will be long-time survivors. Re-irradiation and chemo-radiotherapy using IMRT technique are feasible options decreasing the risk of medullary toxicity.

Miura and collaborators have obtained a dose reduction for brainstem and brain by using half-arc VMAT radiotherapy. Similar results were obtained in the case presented for half-arc VMAT method. 3D and IMRT technique still offers best dose solutions for the brainstem but 2 half-arcs VMAT method offers the lowest mean dose for the brain.

For advanced cases involving large irregularly shaped, requiring elective lymph node irradiation, non-coplanar IMRT and VMAT techniques offers dosimetric advantages but clinical benefits will be validated in the future (Orlandi *et al*., 2014).

5. Conclusions

VMAT technique offers a rapid option with comparable dosimetric results and coverage of the target volume in maxillary sinus cancer. By significantly reducing the dose to the spinal cord compared to 3D-CRT, IMRT can be used in selected cases for dose escalation in order to improve local control. Saving machine-time can be an advantage for choosing VMAT in crowded radiotherapy centers.

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EVALUARE DOZIMETRICĂ COMPARATIVĂ A DIFERITELOR TEHNICI DE RADIOTERAPIE (3D-CRT, IMRT, VMAT) ÎN TRATAMENTUL CANCERELOR RINOSINUSALE

(Rezumat)

Cancerele rinosinusale reprezintă aproximativ 3% din totalul afecțiunilor maligne, majoritatea dezvoltându-se la nivelul sinusurilor maxilare (70%), urmate de etmoid (20%), sinusurile frontale (3%) și sfenoidale (1%). Poziția anatomică și simptomatologia tardivă fac ca aceste tumori maligne să fie dificil de diagnosticat, adesea fiind descoperite în stadii avansate. Abordul chirurgical și tratamentul adjuvant cu radiații are un rol esențial în obținerea controlului local. Organele radiosensibile aflate în proximitate fac dificilă livrarea unor doze de iradiere tumoricidale prin radioterapia convențională. Implementarea tehnologiilor 3D-CRT (3D conformațional), bazate pe utilizarea MLC (colimator multi - lamă) și apoi a tehnicilor de planificare inversă IMRT (terapie cu radiații modulate în intensitate) și VMAT (terapie în arc modulată volumetric) a condus la reducerea dozelor la OAR (organe la risc) și o mai bună omogenitate a dozei în PTV (volumul țintă planificat).

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DOSIMETRIC INFLUENCE OF SYSTEMATIC POSITIONING ERRORS BY INDUCING A 3 mm BIAXIAL SHIFT IN A CASE OF LOCALLY ADVANCED NASOPHARYNX CANCER TREATED WITH EXTERNAL BEAM RADIOTHERAPY

BY

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Abstract. Head & neck malignancy are cancers where radiotherapy is often the main method of treatment especially in advanced cases outdated for surgery. To analyze the dosimetric effects of a biaxial 3 mm position change from isocenter a + 3 mm shift on the X and Y axes was applied. Doses received by OAR (organs at risk) and target volumes treated with sequential boost were evaluated - PTV-T (target volume of the primary tumor) which received 70Gy/35 fractions, PTV-N66 witch received 66Gy/33 fractions and PTV-N50 irradiated with 50Gy/25 fractions. Evaluation of D_{max}, D_{min} and D_{mean} was done both for target volumes and for OAR’s before and after applying the biaxial shift for 3D-CRT(3D-conformal) plans and IMRT (intensity modulated radiation therapy) and VMAT (volumetric modulated arc therapy) alternative plans. The

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dose-volume effect was significant only for phase II and phase III. In case of some OAR's for 3D-CRT technique the maximum recommended dose was exceeded.

Keywords: radiotherapy; IMRT; VMAT; OAR.

1. Introduction

Head & neck malignancies are cancers for which radiotherapy is one of the main methods of treatment especially in advanced cases when surgical approach is impossible. For advanced nasopharyngeal cancer, surgical resection is almost impossible, concurrent radio-chemotherapy being the standard treatment. Surgery remains reserved for selective neck dissection in cases of persistent or recurrent nodal disease. High toxicity is one of the problems associated to conventional radiotherapy. IMRT technique provides better OAR protection (Fig. 1).

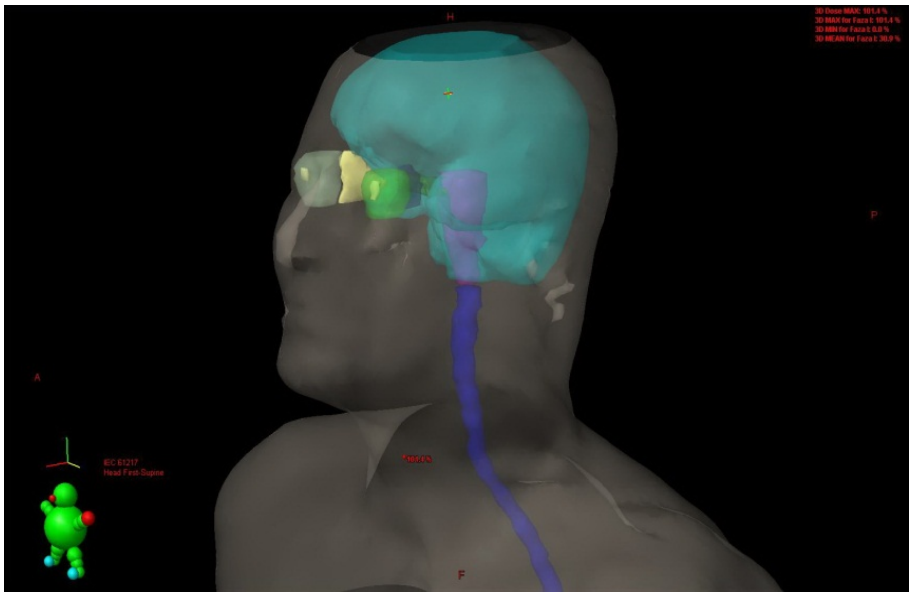

Fig. 1 – 3D reconstruction of organs at risk (OAR)
in nasopharynx radiotherapy.

Associated with a high coverage of target volume (Fig. 2) and a higher dose gradient. The presence of volumes receiving high doses in the immediate vicinity of protected tissues involves an increased risk of errors.

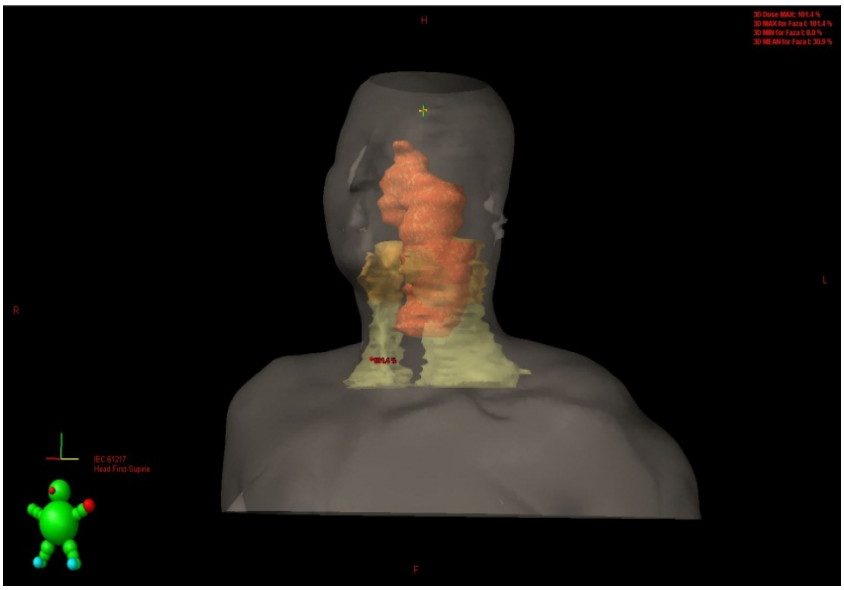

Fig. 2 – 3D reconstruction of target volumes PTV-T (red), PTV-N66 (magenta), PTV-N50 (Yellow).

Usage of CT simulation, orthogonal kV imaging systems for patient positioning and thermoplastic mask decreases the risk of random errors between each fraction. A calibration error of the treatment table or positioning lasers can induce a systematic error with unpredictable consequences for the treatment. To analyze the consequences of such an error a + 3 mm shift on X and Y axes was introduced, then recalculating being executed without 3D-CRT, IMRT and VMAT plans optimization. Dosimetric parameters D_{max}, D_{min} and D_{mean} for target volumes (each phase) and OARs were analyzed comparatively in absolute and relative values (Hong *et al*., 2005; Park and Park, 2016; Yan *et al*., 2013; Iancu and Iancu, 2004).

2. Results

For all techniques 3D-CRT, IMRT, VMAT significant decrease of D_{min} (29.55%, 21.62%, 27.20%) for the phase III of the sequential boost treatment plan is observed in case of 3 mm biaxial shift application to isocenter. In the case of absolute dose delivered by IMRT technique, lower D_{min} value associated with shift effect increases the risk of “cold spots”. The same phenomenon can be observed in the case of phase II, the minimum dose in phase I being less influenced in all situations (see Table 1).

Table 1

Relative Variation of D_{min}, D_{max} and D_{mean} Received by the Target Volumes and Organs at Risk by Applying a + 3 mm Biaxially Isocentric Shift

D_{MIN}	Radiotherapy Technique (Absolute Dose)						Radiotherapy Technique (Dose Change)		
Phases and OARs	3D-CRT(cGy)	3D-CRT-SHIFT(cGy)	IMRT(cGy)	IMRT-SHIFT(cGy)	VMAT(cGy)	VMAT-SHIFT(cGy)	3D-CRT (%)	IMRT (%)	VMAT (%)
Phase III	5725.00	4033.30	3734.90	2927.40	6295.20	4583.00	-29.55	-21.62	-27.20
Phase II	3746.10	3400.40	3288.80	2583.70	5138.70	3931.90	-9.23	-21.44	-23.48
Phase I	2860.20	3106.80	5927.00	4926.80	3008.10	3060.00	8.62	-16.88	1.73
Left parotid	3345.40	3370.80	1390.40	1548.40	2589.70	2685.30	0.76	11.36	3.69
Right parotid	2824.90	2737.60	2095.10	1995.90	2517.40	2312.60	-3.09	-4.73	-8.14
Brain	27.30	26.20	13.00	12.10	15.30	14.30	-4.03	-6.92	-6.54
Brain stem	200.70	183.90	167.80	150.90	264.90	210.30	-8.37	-10.07	-20.61
Spinal cord	22.70	23.90	8.90	9.90	12.90	14.20	5.29	11.24	10.08
Left optic nerve	190.90	179.10	144.80	134.60	367.20	318.20	-6.18	-7.04	-13.34
Optic chiasma	293.80	262.90	246.00	216.10	160.20	146.10	-10.52	-12.15	-8.80
Right optic nerve	182.00	168.80	129.10	117.70	148.60	132.30	-7.25	-8.83	-10.97
Left lens	184.10	173.20	147.00	138.30	163.90	151.90	-5.92	-5.92	-7.32
Right lens	173.90	162.80	119.50	110.70	135.20	123.90	-6.38	-7.36	-8.36

D_{MAX}	Radiotherapy Technique (Absolute Dose)						Radiotherapy Technique (Dose Change)		
Phases and OARs	3D-CRT(cGy)	3D-CRT-SHIFT(cGy)	IMRT(cGy)	IMRT-SHIFT(cGy)	VMAT(cGy)	VMAT-SHIFT(cGy)	3D-CRT (%)	IMRT (%)	VMAT (%)
Phase III	7564.70	7690.90	7441.00	7566.80	7427.70	7590.00	1.67	1.69	2.19
Phase II	7564.70	7690.90	7441.00	7566.80	7427.70	7590.00	1.67	1.69	2.19
Phase I	7564.70	7690.90	7416.90	7560.40	7427.70	7590.00	1.67	1.93	2.19
Left parotid	7242.80	7548.20	7326.30	7478.60	7077.20	7291.50	4.22	2.08	3.03
Right parotid	7177.20	7205.00	7218.30	7317.00	7047.60	7009.80	0.39	1.37	-0.54
Brain	6919.20	6642.30	6128.80	5901.40	6269.60	6229.90	-4.00	-3.71	-0.63
Brain stem	6518.70	6476.30	5498.80	5451.10	5609.30	5589.20	-0.65	-0.87	-0.36
Spinal cord	4498.00	5118.20	4364.40	4547.50	4254.40	4566.30	13.79	4.20	7.33
Left optic nerve	447.90	402.10	414.70	371.60	600.70	538.90	-10.23	-10.39	-10.29
Optic chiasma	444.60	381.80	402.90	338.60	509.10	454.30	-14.13	-15.96	-10.76
Right optic nerve	416.90	365.40	333.40	289.10	476.20	411.00	-12.35	-13.29	-13.69
Left lens	215.90	204.70	176.00	165.90	202.60	187.30	-5.19	-5.74	-7.55
Right lens	202.00	188.70	145.10	135.40	171.50	155.40	-6.58	-6.69	-9.39

D_{MEAN}	Radiotherapy Technique (Absolute Dose)						Radiotherapy Technique (Dose Change)		
Phases and OARs	3D-CRT(cGy)	3D-CRT-SHIFT(cGy)	IMRT(cGy)	IMRT-SHIFT(cGy)	VMAT(cGy)	VMAT-SHIFT(cGy)	3D-CRT (%)	IMRT (%)	VMAT (%)
Phase III	7094.30	7100.50	6798.70	6831.00	7021.10	7034.80	0.09	0.48	0.20
Phase II	6902.90	6932.60	6213.90	6248.50	6803.80	6835.80	0.43	0.56	0.47
Phase I	6282.00	6316.20	7004.50	7039.60	6217.10	6251.50	0.54	0.50	0.55
Left parotid	5837.20	5913.00	4386.80	4624.40	4577.70	4800.60	1.30	5.42	4.87
Right parotid	5794.90	5348.40	4573.10	4156.40	4943.30	4439.80	-7.71	-9.11	-10.19
Brain	318.30	273.10	315.50	266.70	349.40	303.30	-14.20	-15.47	-13.19
Brain stem	1654.90	1443.60	1585.80	1361.50	1736.30	1525.30	-12.77	-14.14	-12.15
Spinal cord	2336.10	2423.30	2129.90	2217.90	1976.40	2078.60	3.73	4.13	5.17
Left optic nerve	304.40	279.20	258.50	235.00	473.80	417.60	-8.28	-9.09	-11.86
Optic chiasma	362.70	318.60	315.80	273.00	318.20	287.80	-12.16	-13.55	-9.55
Right optic nerve	294.90	266.30	231.20	206.90	305.00	268.40	-9.70	-10.51	-12.00
Left lens	200.10	188.60	160.30	151.10	182.20	168.30	-5.75	-5.74	-7.63
Right lens	188.60	176.30	133.30	124.10	154.20	140.90	-6.52	-6.90	-8.63

D_{max} is less modified (minor increase) for all techniques without significant predictable clinical consequences. Also D_{mean} variations are insignificant for phase III using inverse planning techniques compared to 3D-CRT technique. For phases II and III D_{mean} increases are approximately equal in all situations by applying isocenter shift (about 0.5%) (see Table 1).

The consequences of applying biaxial shift for OAR is the D_{max} decrease in most organs excepting left parotid and spinal cord. For all techniques 3D-CRT, IMRT, VMAT left parotid D_{max} increases with 4.22%, 2.08%, 3.03% and D_{mean} increases with 1.30%, 5.42% and 4.87%. For spinal cord D_{max} increases with 13.79%, 4.20% and 7.33%, D_{mean} increases with 3.73%, 4.13% and 5.17%, but only for 3D-CRT plan (D_{max} = 51.18Gy) the absolute dose exceeds upper limit recommendation of Quantec (see Table 1).

3. Discussion

The use of modern radiotherapy methods has reduced the volume exposed to large doses of radiation therapy, improving treatment accuracy, reducing normal tissue toxicity related to irradiation, increased importance given to accurate position verification and correction before delivering radiotherapy. IGRT enables evaluation of geometry for treatment delivery providing a method by which deviations from the original plan of anatomy are determined and this information is used to correct the dosimetric parameters. Bony landmarks were easy to detect and correct and the table shifts for correction of setup deviations could be automatically calculated. An error in radiotherapy is any deviation from intended or planned treatment (Hong *et al*., 2005; Thilmann *et al*., 2006; Dawson and Jaffray, 2007).

The risk of a systematic error is low but the clinical consequences can be unpredictable if the error is not corrected before or during treatment. Decrease of D_{min} in phase III corresponds to target volumes that will receive the entire dose of 70Gy/35 fractions increases the number of cold spots associated with risk of under-dosage in primary tumor volume. The association between a D_{min} decreased in absolute and relative decrease of D_{min} by applying "simulated error", the presence of "cold spots" in a radio-resistant hypoxic zone may be a factor associated with the presence of a residual tumor at the end of treatment. In this case IMRT technique is associated with a higher risk of under-dosage for target volume of primary tumors of the nasopharynx than 3D-CRT and VMAT techniques. By applying the biaxial isocenter shift laterocervical nodal levels (PTV-N66) shows a lower risk of under-dosage than (PTV-T) and the dose effect to supraclavicular nodal (PTV-N50) level is insignificant (Fig. 2). The presence of clinically detectable lymph nodes with a good response to therapy or a significant patient weight loss resulting in neck circumference reduction associated with isocentric shift can bring the skin in the build-up dose area, especially for the case of IMRT technique with more tangential fields (Iancu and Iancu, 2004; Kaur *et al*., 2016; Liu *et al*., 2016).

4. Conclusions

A systematic error of + 3 mm biaxial shift applied to isocenter has no severe consequences on the quality of treatment of nasopharyngeal primary tumor but may result in under-dosage in laterocervical nodal volumes. Adding a random error to the induced systematic error can amplify or reduce the dosimetric effects. In the case of exceeding the value of the total error beyond

the distance limit that manifests intense dose gradient for IMRT and VMAT methods there is a major possibility to irradiation with major dosimetric consequences for the target volumes and normal tissue. Immobilization systems (thermoplastic masks), IG systems and an accurate calibration of the treatment table and positioning lasers ensure the quality of treatment.

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EFECTUL DOZIMETRIC AL ERORILOR SISTEMATICE DE POZIȚIONARE PRIN INDUCEREA ARTIFICIALĂ A UNEI DEPLASĂRI BIAXIALE DE 3 mm A MESEI DE TRATAMENT ÎN RADIOTERAPIA EXTERNĂ A CANCERULUI DE RINOFARINGE LOCAL AVANSAT

(Rezumat)

Cancerele sferei ORL sunt patologii în care radioterapia este de multe ori metoda principală de tratament, în special în cazurile avansate, depășite chirurgical. Analizăm efectul dozimetric a unei deplasări biaxile, de 3 mm, asupra izocentrului, aplicând un shift pe axele x și y de + 3 mm și evaluând dozele la organele de risc și în volumele țintă PTV-T (volumul țintă al tumorii primare), care a primit o doză de 70Gy

în 35 fracțiuni (faza I), PTV- N66 (faza II) și PTV-N50 (faza III) pentru ariile gaglionare laterocervicale și supraclaviculare iradiate cu 66, respectiv 50Gy în 33 și 25 fracțiuni, prin tehnica boost-ului secvențial. Evaluarea parametrilor dozimetrici Dmax, Dmean și Dmin s-a făcut atât pentru volumele țintă cât și pentru OAR (organele de risc) înainte și după aplicarea shiftului pentru planurile 3D-CRT (3D conformațional) și pe planurile alternative IMRT (radioterapie cu intensitate modulată) și VMAT (radioterapie rotațională cu intensitate modulată). Efectul asupra volumelor țintă ca distribuție a dozei a fost semnificativ doar în fazele II și III. În cazul OAR, prin tehnica 3D, în urma shiftului s-a depășit doza maximă recomandată de ghidul dozimetric QUANTEC.